

5 Polynomials and Polynomial Functions

PA

M11.A.2.2.2

M11.D.2.2.1

M11.D.2.2.2

M11.A.3.2.1

- 5.1 Use Properties of Exponents
- 5.2 Evaluate and Graph Polynomial Functions
- 5.3 Add, Subtract, and Multiply Polynomials
- 5.4 Factor and Solve Polynomial Equations
- 5.5 Apply the Remainder and Factor Theorems
- 5.6 Find Rational Zeros
- 5.7 Apply the Fundamental Theorem of Algebra
- 5.8 Analyze Graphs of Polynomial Functions
- 5.9 Write Polynomial Functions and Models

Before

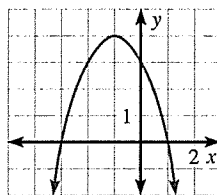
In previous chapters, you learned the following skills, which you'll use in Chapter 5: graphing functions, factoring, and solving equations.

Prerequisite Skills

VOCABULARY CHECK

Copy and complete the statement.

- The **zeros** of the function graphed are ?.
- The **maximum value** of the function graphed is ?.
- The **standard form** of a quadratic equation in one variable is ? where $a \neq 0$.



SKILLS CHECK

Graph the function. Label the vertex and the axis of symmetry.

(Review pp. 236, 245 for 5.2.)

- $y = -2(x - 1)^2 + 4$
- $y = 3(x - 2)(x + 3)$
- $y = -x^2 - 4x + 4$

Factor the expression. (Review pp. 252, 259 for 5.4.)

- $x^2 + 9x + 20$
- $2x^2 + 5x - 3$
- $9x^2 - 64$

Solve the equation. (Review pp. 252, 259 for 5.4–5.7.)

- $2x^2 + x + 6 = 0$
- $10x^2 + 13x = 3$
- $x^2 + 6x + 2 = 20$

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MULTIPLE CHOICE

9. What is the value of k in the equation $6x^2 - 11x - 10 = (3x + 2)(2x - k)$?
- A -8 B -5
C 5 D 8
10. What is the real part of the standard form of the expression $(5 + i)(10 - i)$?
- A 49 B 50
C 51 D 54
11. For what value of c is $x^2 - 7x + c$ a perfect square trinomial?
- A $\frac{7}{2}$
B $\frac{49}{4}$
C $\frac{49}{2}$
D 49
12. What is the maximum value of the function $y = -3(x - 2)^2 + 6$?
- A -6 B 2
C 6 D There is no maximum value.
13. What is the greatest zero of the function $y = x^2 - 25x + 66$?
- A -22
B -3
C 3
D 22
14. What is the absolute value of $-5 + 12i$?
- A 5
B 7
C 13
D 17

OPEN-ENDED

15. A parabola passes through the following points:
(0, -22), (2, -6), (5, -12)
- A. What is the x -coordinate of the vertex of the parabola?
B. Use your calculator to write a quadratic equation that models the given data.
16. Given the function $f(x) = 4x^2 + 24x + 39$ —
- A. Find the y -intercept of the function.
B. Find the minimum value of the function.
17. A volleyball is hit upward by a player in a game. The height h (in feet) of the volleyball after t seconds is given by the function $h = -16t^2 + 30t + 6$.
- A. What is the maximum height of the volleyball? *Explain* your reasoning.
B. After how many seconds does the volleyball reach its maximum height?
C. After how many seconds does the volleyball hit the ground?

Now

In Chapter 5, you will apply the big ideas listed below and reviewed in the Chapter Summary on page 401. You will also use the key vocabulary listed below.

Big Ideas

- ① Graphing polynomial functions
- ② Performing operations with polynomials
- ③ Solving polynomial equations and finding zeros

KEY VOCABULARY

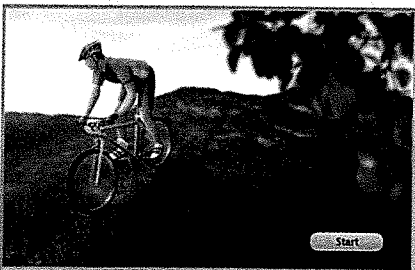
- polynomial, p. 337
- polynomial function, p. 337
- synthetic substitution, p. 338
- end behavior, p. 339
- factored completely, p. 353
- factor by grouping, p. 354
- quadratic form, p. 355
- polynomial long division, p. 362
- synthetic division, p. 363
- repeated solution, p. 379
- local maximum, p. 388
- local minimum, p. 388
- finite differences, p. 393

Why?

You can use polynomial functions to model real-life situations. For example, you can use a polynomial function to model the relationship between the speed of an object and the power needed to maintain that speed.

Animated Algebra

The animation illustrated below for Exercise 61 on page 351 helps you answer this question: How does the power needed to keep a bicycle moving at a constant speed change as the conditions change?



The power exerted by a bicyclist depends on speed and resistance.

Settings

Bicyclist's Speed $\leftarrow$ 5 mph

Road Surface $\leftarrow$ 2% Incline

Wind Speed $\leftarrow$ 5 mph

Calculations

$P =$

$F =$

Power needed =



So far, we have looked at a bicyclist traveling on level ground. The power equation will change depending on the amount of resistance.

Use the sliders to see how the road slope and wind speed affect the resistance.

Animated Algebra at classzone.com

Other animations for Chapter 5: pages 331, 340, 371, 388, 396, and 401

5.1 Use Properties of Exponents

PA

M11.A.2.2.2 Simplify/evaluate expressions involving multiplying with exponents... powers of powers... and powers of products...

Before

You evaluated powers.

Now

You will simplify expressions involving powers.

Why?

So you can compare the volumes of two stars, as in Example 5.



Key Vocabulary
• scientific notation

Consider what happens when you multiply two powers that have the same base:

$$2^3 \cdot 2^5 = (2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2 \cdot 2 \cdot 2) = 2^8$$

Note that the exponent 8 in the product is the sum of the exponents 3 and 5 in the factors. This property is one of several properties of exponents shown below.

KEY CONCEPT

For Your Notebook

Properties of Exponents

Let a and b be real numbers and let m and n be integers.

Property Name	Definition	Example
Product of Powers	$a^m \cdot a^n = a^{m+n}$	$5^3 \cdot 5^{-1} = 5^{3+(-1)} = 5^2 = 25$
Power of a Power	$(a^m)^n = a^{mn}$	$(3^3)^2 = 3^{3 \cdot 2} = 3^6 = 729$
Power of a Product	$(ab)^m = a^m b^m$	$(2 \cdot 3)^4 = 2^4 \cdot 3^4 = 1296$
Negative Exponent	$a^{-m} = \frac{1}{a^m}, a \neq 0$	$7^{-2} = \frac{1}{7^2} = \frac{1}{49}$
Zero Exponent	$a^0 = 1, a \neq 0$	$(-89)^0 = 1$
Quotient of Powers	$\frac{a^m}{a^n} = a^{m-n}, a \neq 0$	$\frac{6^{-3}}{6^{-6}} = 6^{-3-(-6)} = 6^3 = 216$
Power of a Quotient	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$	$\left(\frac{4}{7}\right)^2 = \frac{4^2}{7^2} = \frac{16}{49}$

AVOID ERRORS

When you multiply powers, do not multiply the bases. For example, $3^2 \cdot 3^5 \neq 9^7$.

EXAMPLE 1 Evaluate numerical expressions

- a. $(-4 \cdot 2^5)^2 = (-4)^2 \cdot (2^5)^2$ **Power of a product property**
 $= 16 \cdot 2^{5 \cdot 2}$ **Power of a power property**
 $= 16 \cdot 2^{10} = 16,384$ **Simplify and evaluate power.**
- b. $\left(\frac{11^5}{11^8}\right)^{-1} = \frac{11^8}{11^5}$ **Negative exponent property**
 $= 11^{8-5}$ **Quotient of powers property**
 $= 11^3 = 1331$ **Simplify and evaluate power.**

SCIENTIFIC NOTATION A number is expressed in **scientific notation** if it is in the form $c \times 10^n$ where $1 \leq c < 10$ and n is an integer. When you work with numbers in scientific notation, the properties of exponents can make calculations easier.

EXAMPLE 2 Use scientific notation in real life

LOCUSTS A swarm of locusts may contain as many as 85 million locusts per square kilometer and cover an area of 1200 square kilometers. About how many locusts are in such a swarm?

Solution

Number of locusts	=	Locusts per square kilometer	×	Number of square kilometers	
	=	85,000,000	×	1200	Substitute values.
	=	(8.5×10^7)	(1.2×10^3)		Write in scientific notation.
	=	(8.5×1.2)	$(10^7 \times 10^3)$		Use multiplication properties.
	=	10.2×10^{10}			Product of powers property
	=	1.02×10^1	$\times 10^{10}$		Write 10.2 in scientific notation.
	=	1.02×10^{11}			Product of powers property

► The number of locusts is about 1.02×10^{11} , or about 102,000,000,000.



REVIEW SCIENTIFIC NOTATION

For help with scientific notation, see p. 982.

GUIDED PRACTICE for Examples 1 and 2

Evaluate the expression. Tell which properties of exponents you used.

1. $(4^2)^3$

2. $(-8)(-8)^3$

3. $\left(\frac{2}{9}\right)^3$

4. $\frac{6 \cdot 10^{-4}}{9 \cdot 10^7}$

SIMPLIFYING EXPRESSIONS You can use the properties of exponents to simplify algebraic expressions. A simplified expression contains only positive exponents.

EXAMPLE 3 Simplify expressions

a. $b^{-4}b^6b^7 = b^{-4+6+7} = b^9$ Product of powers property

b. $\left(\frac{r^{-2}}{s^3}\right)^{-3} = \frac{(r^{-2})^{-3}}{(s^3)^{-3}}$ Power of a quotient property

$= \frac{r^6}{s^{-9}}$ Power of a power property

$= r^6s^9$ Negative exponent property

c. $\frac{16m^4n^{-5}}{2n^{-5}} = 8m^4n^{-5-(-5)}$ Quotient of powers property

$= 8m^4n^0 = 8m^4$ Zero exponent property

INTERPRET BASES

In this book, it is assumed that any base with a zero or negative exponent is nonzero.

Animated Algebra at classzone.com



EXAMPLE 4 Standardized Test Practice

What is the simplified form of $\frac{(x^{-3}y^3)^2}{x^5y^6}$?

- (A) x^{11} (B) $\frac{1}{x^{11}}$ (C) $\frac{1}{x^6y}$ (D) $\frac{1}{x^{11}y}$

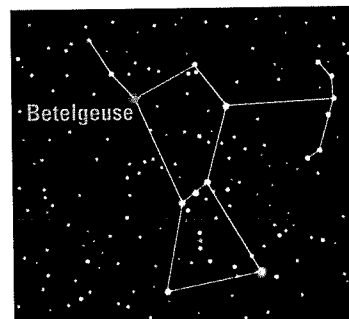
Solution

$$\begin{aligned}\frac{(x^{-3}y^3)^2}{x^5y^6} &= \frac{(x^{-3})^2(y^3)^2}{x^5y^6} && \text{Power of a product property} \\ &= \frac{x^{-6}y^6}{x^5y^6} && \text{Power of a power property} \\ &= x^{-6-5}y^{6-6} && \text{Quotient of powers property} \\ &= x^{-11}y^0 && \text{Simplify exponents.} \\ &= x^{-11} \cdot 1 && \text{Zero exponent property} \\ &= \frac{1}{x^{11}} && \text{Negative exponent property}\end{aligned}$$

► The correct answer is B. (A) (B) (C) (D)

EXAMPLE 5 Compare real-life volumes

ASTRONOMY Betelgeuse is one of the stars found in the constellation Orion. Its radius is about 1500 times the radius of the sun. How many times as great as the sun's volume is Betelgeuse's volume?



Solution

Let r represent the sun's radius. Then $1500r$ represents Betelgeuse's radius.

$$\begin{aligned}\frac{\text{Betelgeuse's volume}}{\text{Sun's volume}} &= \frac{\frac{4}{3}\pi(1500r)^3}{\frac{4}{3}\pi r^3} && \text{The volume of a sphere is } \frac{4}{3}\pi r^3. \\ &= \frac{\cancel{\frac{4}{3}}\pi\cancel{1500^3}r^3}{\cancel{\frac{4}{3}}\pi r^3} && \text{Power of a product property} \\ &= 1500^3 r^0 && \text{Quotient of powers property} \\ &= 1500^3 \cdot 1 && \text{Zero exponent property} \\ &= 3,375,000,000 && \text{Evaluate power.}\end{aligned}$$

► Betelgeuse's volume is about 3.4 billion times as great as the sun's volume.

**GUIDED PRACTICE** for Examples 3, 4, and 5

Simplify the expression. Tell which properties of exponents you used.

5. $x^{-6}x^5x^3$

6. $(7y^2z^5)(y^{-4}z^{-1})$

7. $\left(\frac{s^3}{t^{-4}}\right)^2$

8. $\left(\frac{x^4y^{-2}}{x^3y^6}\right)^3$

5.1 EXERCISES**HOMEWORK KEY**○ = **WORKED-OUT SOLUTIONS**
on p. WS9 for Exs. 17, 31, and 51★ = **STANDARDIZED TEST PRACTICE**
Exs. 2, 36, 46, 51, and 53**SKILL PRACTICE**1. **VOCABULARY** State the name of the property illustrated.

a. $a^m \cdot a^n = a^{m+n}$

b. $a^{-m} = \frac{1}{a^m}, a \neq 0$

c. $(ab)^m = a^mb^m$

2. ★ **WRITING** Is the number 25.2×10^{-3} in scientific notation? *Explain.***EXAMPLE 1**on p. 330
for Exs. 3–14**EVALUATING NUMERICAL EXPRESSIONS** Evaluate the expression. Tell which properties of exponents you used.

3. $3^3 \cdot 3^2$

4. $(4^{-2})^3$

5. $(-5)(-5)^4$

6. $(2^4)^2$

7. $\frac{5^2}{5^5}$

8. $\left(\frac{3}{5}\right)^4$

9. $\left(\frac{2}{7}\right)^{-3}$

10. $9^3 \cdot 9^{-1}$

11. $\frac{3^4}{3^{-2}}$

12. $\left(\frac{2}{3}\right)^{-5}\left(\frac{2}{3}\right)^4$

13. $6^3 \cdot 6^0 \cdot 6^{-5}$

14. $\left(\left(\frac{1}{2}\right)^{-5}\right)^2$

EXAMPLE 2on p. 331
for Exs. 15–23**SCIENTIFIC NOTATION** Write the answer in scientific notation.

15. $(4.2 \times 10^3)(1.5 \times 10^6)$

16. $(1.2 \times 10^{-3})(6.7 \times 10^{-7})$

17. $(6.3 \times 10^5)(8.9 \times 10^{-12})$

18. $(7.2 \times 10^9)(9.4 \times 10^8)$

19. $(2.1 \times 10^{-4})^3$

20. $(4.0 \times 10^3)^4$

21. $\frac{8.1 \times 10^{12}}{5.4 \times 10^9}$

22. $\frac{1.1 \times 10^{-3}}{5.5 \times 10^{-8}}$

23. $\frac{(7.5 \times 10^8)(4.5 \times 10^{-4})}{1.5 \times 10^7}$

EXAMPLES 3 and 4on pp. 331–332
for Exs. 24–39**SIMPLIFYING ALGEBRAIC EXPRESSIONS** Simplify the expression. Tell which properties of exponents you used.

24. $\frac{w^{-2}}{w^6}$

25. $(2^2y^3)^5$

26. $(p^3q^2)^{-1}$

27. $(w^3x^{-2})(w^6x^{-1})$

28. $(5s^{-2}t^4)^{-3}$

29. $(3a^3b^5)^{-3}$

30. $\frac{x^{-1}y^2}{x^2y^{-1}}$

31. $\frac{3c^3d}{9cd^{-1}}$

32. $\frac{4r^4s^5}{24r^4s^{-5}}$

33. $\frac{2a^3b^{-4}}{3a^5b^{-2}}$

34. $\frac{y^{11}}{4z^3} \cdot \frac{8z^7}{y^7}$

35. $\frac{x^2y^{-3}}{3y^2} \cdot \frac{y^2}{x^{-4}}$

36. ★ **MULTIPLE CHOICE** What is the simplified form of $\frac{2x^2y}{6xy^{-1}}$?

Ⓐ $\frac{y^2}{3}$

Ⓑ $\frac{xy^2}{3}$

Ⓒ $\frac{x}{3}$

Ⓓ $\frac{1}{3}$

ERROR ANALYSIS Describe and correct the error in simplifying the expression.

37.

$$\frac{x^{10}}{x^2} = x^5 \quad \times$$

38.

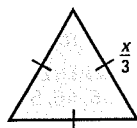
$$x^5 \cdot x^3 = x^{15} \quad \times$$

39.

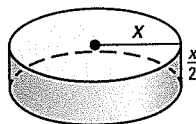
$$(-3)^2(-3)^4 = 9^6 \quad \times$$

GEOMETRY Write an expression for the figure's area or volume in terms of x .

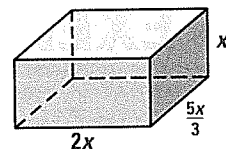
40. $A = \frac{\sqrt{3}}{4}s^2$



41. $V = \pi r^2 h$



42. $V = \ell wh$



REASONING Write an expression that makes the statement true.

43. $x^{15}y^{12}z^8 = x^4y^7z^{11} \cdot ?$

44. $3x^3y^2 = \frac{12x^2y^5}{?}$

45. $(a^5b^4)^2 = a^{14}b^{-1} \cdot ?$

46. **★ OPEN-ENDED MATH** Find three different ways to complete the following statement so that it is true: $x^{12}y^{16} = (x^?y^?)(x^?y^?)$.

CHALLENGE Refer to the properties of exponents on page 330.

47. Show how the negative exponent property can be derived from the quotient of powers property and the zero exponent property.
48. Show how the quotient of powers property can be derived from the product of powers property and the negative exponent property.

PROBLEM SOLVING

EXAMPLE 2

on p. 331
for Exs. 49–50

49. **OCEAN VOLUME** The table shows the surface areas and average depths of four oceans. Calculate the volume of each ocean by multiplying the surface area of each ocean by its average depth. Write your answers in scientific notation.

Ocean	Surface area (square meters)	Average depth (meters)
Pacific	1.56×10^{14}	4.03×10^3
Atlantic	7.68×10^{13}	3.93×10^3
Indian	6.86×10^{13}	3.96×10^3
Arctic	1.41×10^{13}	1.21×10^3



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50. **EARTH SCIENCE** The continents of Earth move at a very slow rate. The South American continent has been moving about 0.000022 mile per year for the past 125,000,000 years. How far has the continent moved in that time? Write your answer in scientific notation.

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EXAMPLE 5

on p. 332
for Exs. 51–52

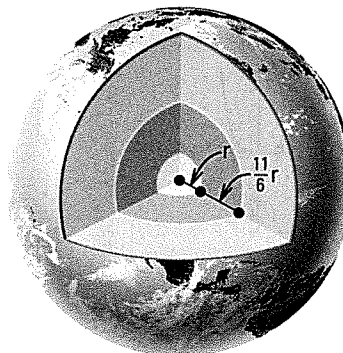
51. ★ **SHORT RESPONSE** A typical cultured black pearl is made by placing a bead with a diameter of 6 millimeters inside an oyster. The resulting pearl has a diameter of about 9 millimeters. *Compare* the volume of the resulting pearl with the volume of the bead.

52. **MULTI-STEP PROBLEM** A can of tennis balls consists of three spheres of radius r stacked vertically inside a cylinder of radius r and height h .
- Write an expression for the total volume of the three tennis balls in terms of r .
 - Write an expression for the volume of the cylinder in terms of r and h .
 - Write an expression for h in terms of r using the fact that the height of the cylinder is the sum of the diameters of the three tennis balls.
 - What fraction of the can's volume is taken up by the tennis balls?

53. ★ **EXTENDED RESPONSE** You can think of a penny as a cylinder with a radius of about 9.53 millimeters and a height of about 1.55 millimeters.

- Calculate** Approximate the volume of a penny. Give your answer in cubic meters.
- Estimate** Approximate the volume of your classroom in cubic meters. *Explain* how you obtained your answer.
- Interpret** Use your results from parts (a) and (b) to estimate how many pennies it would take to fill your classroom. Do you think your answer is an overestimate or an underestimate? *Explain*.

54. **CHALLENGE** Earth's core is approximately spherical in shape and is divided into a solid inner core (the yellow region in the diagram shown) and a liquid outer core (the dark orange region in the diagram).



- Earth's radius is about 5 times as great as the radius of Earth's inner core. Find the ratio of Earth's total volume to the volume of Earth's inner core.
- Find the ratio of the volume of Earth's outer core to the volume of Earth's inner core.

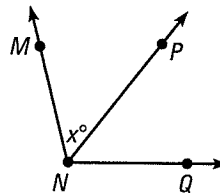
PA**PENNSYLVANIA MIXED REVIEW****TEST PRACTICE** at classzone.com

55. What are the zeros of the function $y = 2x^2 + 5x - 12$?

- (A) $-\frac{3}{2}, -4$ (B) $-\frac{3}{2}, 4$ (C) $\frac{3}{2}, -4$ (D) $\frac{3}{2}, 4$

56. In the diagram, $\overrightarrow{NP}$ bisects $\angle MNQ$ and $m\angle MNP$ is x° . Which equation can be used to find y , which represents $m\angle MNQ$?

- (A) $y = \frac{x}{2}$ (B) $y = x$
(C) $y = 2x$ (D) $y = 180 - x$



5.2 End Behavior of Polynomial Functions

MATERIALS • graphing calculator

QUESTION How is the end behavior of a polynomial function related to the function's equation?

Functions of the form $f(x) = \pm x^n$, where n is a positive integer, are examples of *polynomial functions*. The *end behavior* of a polynomial function's graph is its behavior as x approaches positive infinity ($+\infty$) or as x approaches negative infinity ($-\infty$).

EXPLORE Investigate the end behavior of $f(x) = \pm x^n$ where n is even

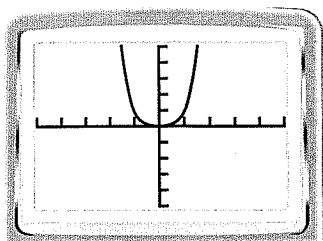
Graph the function. Describe the end behavior of the graph.

a. $f(x) = x^4$

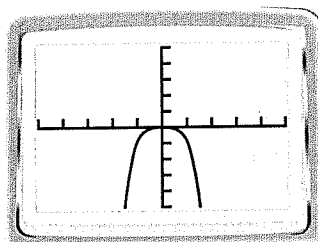
b. $f(x) = -x^4$

STEP 1 *Graph functions* Graph each function on a graphing calculator.

a.



b.



STEP 2 *Describe end behavior* Summarize the end behavior of each function.

Function	As x approaches $-\infty$	As x approaches $+\infty$
a. $f(x) = x^4$	$f(x)$ approaches $+\infty$	$f(x)$ approaches $+\infty$
b. $f(x) = -x^4$	$f(x)$ approaches $-\infty$	$f(x)$ approaches $-\infty$

DRAW CONCLUSIONS Use your observations to complete these exercises

Graph the function. Then describe its end behavior as shown above.

1. $f(x) = x^5$

2. $f(x) = -x^5$

3. $f(x) = x^6$

4. $f(x) = -x^6$

5. Make a conjecture about the end behavior of each family of functions.

a. $f(x) = x^n$ where n is odd

b. $f(x) = -x^n$ where n is odd

c. $f(x) = x^n$ where n is even

d. $f(x) = -x^n$ where n is even

6. Make a conjecture about the end behavior of the function $f(x) = x^6 - x$. Explain your reasoning.

5.2 Evaluate and Graph Polynomial Functions



Before

You evaluated and graphed linear and quadratic functions.

Now

You will evaluate and graph other polynomial functions.

Why?

So you can model skateboarding participation, as in Ex. 55.

Key Vocabulary

- polynomial
- polynomial function
- synthetic substitution
- end behavior

Recall that a monomial is a number, a variable, or a product of numbers and variables. A **polynomial** is a monomial or a sum of monomials.

A **polynomial function** is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n \neq 0$, the exponents are all whole numbers, and the coefficients are all real numbers. For this function, a_n is the **leading coefficient**, n is the **degree**, and a_0 is the **constant term**. A polynomial function is in **standard form** if its terms are written in descending order of exponents from left to right.

Common Polynomial Functions			
Degree	Type	Standard form	Example
0	Constant	$f(x) = a_0$	$f(x) = -14$
1	Linear	$f(x) = a_1 x + a_0$	$f(x) = 5x - 7$
2	Quadratic	$f(x) = a_2 x^2 + a_1 x + a_0$	$f(x) = 2x^2 + x - 9$
3	Cubic	$f(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0$	$f(x) = x^3 - x^2 + 3x$
4	Quartic	$f(x) = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$	$f(x) = x^4 + 2x - 1$

EXAMPLE 1 Identify polynomial functions

Decide whether the function is a polynomial function. If so, write it in standard form and state its degree, type, and leading coefficient.

a. $h(x) = x^4 - \frac{1}{4}x^2 + 3$

b. $g(x) = 7x - \sqrt{3} + \pi x^2$

c. $f(x) = 5x^2 + 3x^{-1} - x$

d. $k(x) = x + 2^x - 0.6x^5$

Solution

- The function is a polynomial function that is already written in standard form. It has degree 4 (quartic) and a leading coefficient of 1.
- The function is a polynomial function written as $g(x) = \pi x^2 + 7x - \sqrt{3}$ in standard form. It has degree 2 (quadratic) and a leading coefficient of π .
- The function is not a polynomial function because the term $3x^{-1}$ has an exponent that is not a whole number.
- The function is not a polynomial function because the term 2^x does not have a variable base and an exponent that is a whole number.

EXAMPLE 2 Evaluate by direct substitution

Use direct substitution to evaluate $f(x) = 2x^4 - 5x^3 - 4x + 8$ when $x = 3$.

$$f(x) = 2x^4 - 5x^3 - 4x + 8 \quad \text{Write original function.}$$

$$f(3) = 2(3)^4 - 5(3)^3 - 4(3) + 8 \quad \text{Substitute 3 for } x.$$

$$= 162 - 135 - 12 + 8 \quad \text{Evaluate powers and multiply.}$$

$$= 23 \quad \text{Simplify.}$$

**GUIDED PRACTICE** for Examples 1 and 2

Decide whether the function is a polynomial function. If so, write it in standard form and state its degree, type, and leading coefficient.

1. $f(x) = 13 - 2x$ 2. $p(x) = 9x^4 - 5x^{-2} + 4$ 3. $h(x) = 6x^2 + \pi - 3x$

Use direct substitution to evaluate the polynomial function for the given value of x .

4. $f(x) = x^4 + 2x^3 + 3x^2 - 7$; $x = -2$ 5. $g(x) = x^3 - 5x^2 + 6x + 1$; $x = 4$

SYNTHETIC SUBSTITUTION Another way to evaluate a polynomial function is to use **synthetic substitution**. This method, shown in the next example, involves fewer operations than direct substitution.

EXAMPLE 3 Evaluate by synthetic substitution

Use synthetic substitution to evaluate $f(x)$ from Example 2 when $x = 3$.

Solution

STEP 1 Write the coefficients of $f(x)$ in order of descending exponents. Write the value at which $f(x)$ is being evaluated to the left.

$$\begin{array}{r|rrrrrr} x\text{-value} \rightarrow 3 & 2 & -5 & 0 & -4 & 8 & \leftarrow \text{coefficients} \end{array}$$

STEP 2 Bring down the leading coefficient. Multiply the leading coefficient by the x -value. Write the product under the second coefficient. Add.

$$\begin{array}{r|rrrrrr} 3 & 2 & -5 & 0 & -4 & 8 \\ & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\ & 2 & 6 & 1 & 3 & 15 \end{array}$$

STEP 3 Multiply the previous sum by the x -value. Write the product under the third coefficient. Add. Repeat for all of the remaining coefficients. The final sum is the value of $f(x)$ at the given x -value.

$$\begin{array}{r|rrrrrr} 3 & 2 & -5 & 0 & -4 & 8 \\ & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\ & 2 & 6 & 3 & 9 & 15 \\ & & \downarrow & \nearrow & \downarrow & \nearrow \\ & & 2 & 1 & 3 & 5 & 23 \end{array}$$

► Synthetic substitution gives $f(3) = 23$, which matches the result in Example 2.

AVOID ERRORS

The row of coefficients for $f(x)$ must include a coefficient of 0 for the "missing" x^2 -term.

END BEHAVIOR The **end behavior** of a function's graph is the behavior of the graph as x approaches positive infinity ($+\infty$) or negative infinity ($-\infty$). For the graph of a polynomial function, the end behavior is determined by the function's degree and the sign of its leading coefficient.

READING

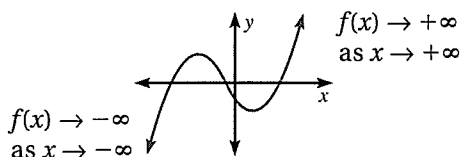
The expression " $x \rightarrow +\infty$ " is read as "x approaches positive infinity."

KEY CONCEPT

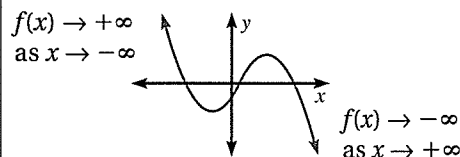
For Your Notebook

End Behavior of Polynomial Functions

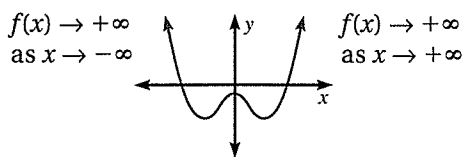
Degree: odd
Leading coefficient: positive



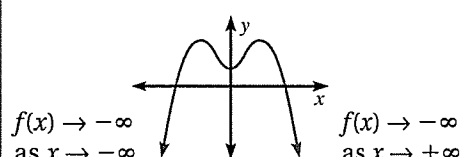
Degree: odd
Leading coefficient: negative



Degree: even
Leading coefficient: positive



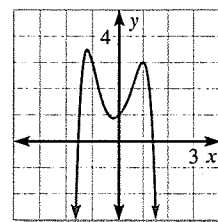
Degree: even
Leading coefficient: negative



EXAMPLE 4 Standardized Test Practice

What is true about the degree and leading coefficient of the polynomial function whose graph is shown?

- (A) Degree is odd; leading coefficient is positive
- (B) Degree is odd; leading coefficient is negative
- (C) Degree is even; leading coefficient is positive
- (D) Degree is even; leading coefficient is negative



From the graph, $f(x) \rightarrow -\infty$ as $x \rightarrow -\infty$ and $f(x) \rightarrow -\infty$ as $x \rightarrow +\infty$. So, the degree is even and the leading coefficient is negative.

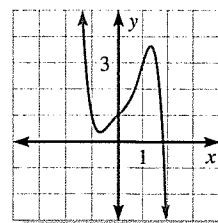
► The correct answer is D. (A) (B) (C) (D)



GUIDED PRACTICE for Examples 3 and 4

Use synthetic substitution to evaluate the polynomial function for the given value of x .

6. $f(x) = 5x^3 + 3x^2 - x + 7$; $x = 2$
7. $g(x) = -2x^4 - x^3 + 4x - 5$; $x = -1$
8. Describe the degree and leading coefficient of the polynomial function whose graph is shown.



GRAPHING POLYNOMIAL FUNCTIONS To graph a polynomial function, first plot points to determine the shape of the graph's middle portion. Then use what you know about end behavior to sketch the ends of the graph.

EXAMPLE 5 Graph polynomial functions

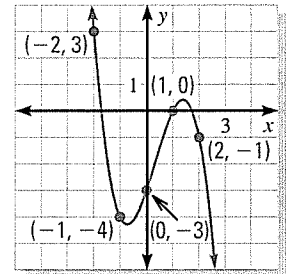
Graph (a) $f(x) = -x^3 + x^2 + 3x - 3$ and (b) $f(x) = x^4 - x^3 - 4x^2 + 4$.

Solution

- a. To graph the function, make a table of values and plot the corresponding points. Connect the points with a smooth curve and check the end behavior.

x	-3	-2	-1	0	1	2	3
y	24	3	-4	-3	0	-1	-12

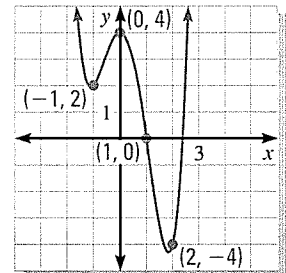
The degree is odd and leading coefficient is negative. So, $f(x) \rightarrow +\infty$ as $x \rightarrow -\infty$ and $f(x) \rightarrow -\infty$ as $x \rightarrow +\infty$.



- b. To graph the function, make a table of values and plot the corresponding points. Connect the points with a smooth curve and check the end behavior.

x	-3	-2	-1	0	1	2	3
y	76	12	2	4	0	-4	22

The degree is even and leading coefficient is positive. So, $f(x) \rightarrow +\infty$ as $x \rightarrow -\infty$ and $f(x) \rightarrow +\infty$ as $x \rightarrow +\infty$.



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EXAMPLE 6 Solve a multi-step problem

PHYSICAL SCIENCE The energy E (in foot-pounds) in each square foot of a wave is given by the model $E = 0.0029s^4$ where s is the wind speed (in knots). Graph the model. Use the graph to estimate the wind speed needed to generate a wave with 1000 foot-pounds of energy per square foot.

Solution

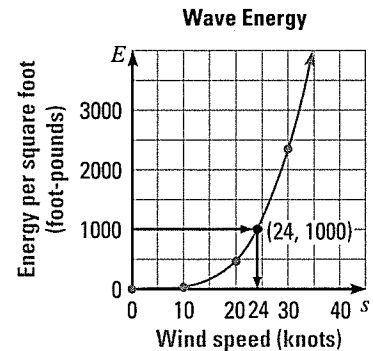
STEP 1 Make a table of values. The model only deals with positive values of s .

s	0	10	20	30	40
E	0	29	464	2349	7424

STEP 2 Plot the points and connect them with a smooth curve. Because the leading coefficient is positive and the degree is even, the graph rises to the right.

STEP 3 Examine the graph to see that $s \approx 24$ when $E = 1000$.

► The wind speed needed to generate the wave is about 24 knots.



**GUIDED PRACTICE** for Examples 5 and 6

Graph the polynomial function.

9. $f(x) = x^4 + 6x^2 - 3$

10. $f(x) = -x^3 + x^2 + x - 1$

11. $f(x) = 4 - 2x^3$

12. **WHAT IF?** If wind speed is measured in miles per hour, the model in Example 6 becomes $E = 0.0051s^4$. Graph this model. What wind speed is needed to generate a wave with 2000 foot-pounds of energy per square foot?

5.2 EXERCISES**HOMEWORK KEY**○ = **WORKED-OUT SOLUTIONS**
on p. WS10 for Exs. 21, 27, and 57★ = **STANDARDIZED TEST PRACTICE**
Exs. 2, 24, 37, 50, 52, and 59◆ = **MULTIPLE REPRESENTATIONS**
Ex. 56**SKILL PRACTICE**

1. **VOCABULARY** Identify the degree, type, leading coefficient, and constant term of the polynomial function $f(x) = 6 + 2x^2 - 5x^4$.

2. ★ **WRITING** Explain what is meant by the end behavior of a polynomial function.

EXAMPLE 1on p. 337
for Exs. 3–8

POLYNOMIAL FUNCTIONS Decide whether the function is a polynomial function. If so, write it in standard form and state its degree, type, and leading coefficient.

3. $f(x) = 8 - x^2$

4. $f(x) = 6x + 8x^4 - 3$

5. $g(x) = \pi x^4 + \sqrt{6}$

6. $h(x) = x^3\sqrt{10} + 5x^{-2} + 1$

7. $h(x) = -\frac{5}{2}x^3 + 3x - 10$

8. $g(x) = 8x^3 - 4x^2 + \frac{2}{x}$

EXAMPLE 2on p. 338
for Exs. 9–14

DIRECT SUBSTITUTION Use direct substitution to evaluate the polynomial function for the given value of x .

9. $f(x) = 5x^3 - 2x^2 + 10x - 15$; $x = -1$

10. $f(x) = 8x + 5x^4 - 3x^2 - x^3$; $x = 2$

11. $g(x) = 4x^3 - 2x^5$; $x = -3$

12. $h(x) = 6x^3 - 25x + 20$; $x = 5$

13. $h(x) = x + \frac{1}{2}x^4 - \frac{3}{4}x^3 + 10$; $x = -4$

14. $g(x) = 4x^5 + 6x^3 + x^2 - 10x + 5$; $x = -2$

EXAMPLE 3on p. 338
for Exs. 15–23

SYNTHETIC SUBSTITUTION Use synthetic substitution to evaluate the polynomial function for the given value of x .

15. $f(x) = 5x^3 - 2x^2 - 8x + 16$; $x = 3$

16. $f(x) = 8x^4 + 12x^3 + 6x^2 - 5x + 9$; $x = -2$

17. $g(x) = x^3 + 8x^2 - 7x + 35$; $x = -6$

18. $h(x) = -8x^3 + 14x - 35$; $x = 4$

19. $f(x) = -2x^4 + 3x^3 - 8x + 13$; $x = 2$

20. $g(x) = 6x^5 + 10x^3 - 27$; $x = -3$

21. $h(x) = -7x^3 + 11x^2 + 4x$; $x = 3$

22. $f(x) = x^4 + 3x - 20$; $x = 4$

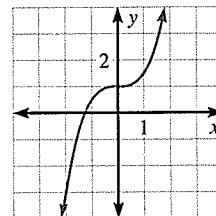
23. **ERROR ANALYSIS** Describe and correct the error in evaluating the polynomial function $f(x) = -4x^4 + 9x^2 - 21x + 7$ when $x = -2$.

-2	-4	9	-21	7
		8	-34	110
	-4	17	-55	117



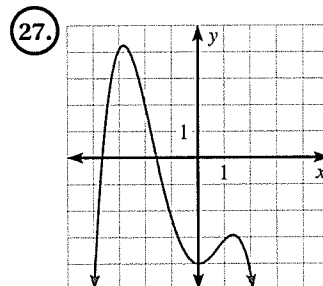
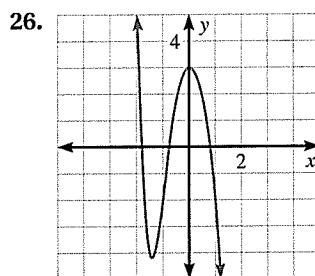
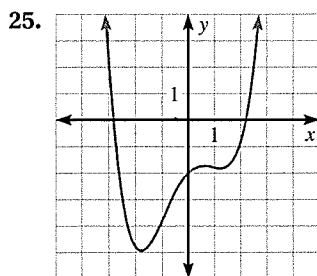
EXAMPLE 4
on p. 339
for Exs. 24–27

24. ★ **MULTIPLE CHOICE** The graph of a polynomial function is shown. What is true about the function's degree and leading coefficient?



- (A) The degree is odd and the leading coefficient is positive.
(B) The degree is odd and the leading coefficient is negative.
(C) The degree is even and the leading coefficient is positive.
(D) The degree is even and the leading coefficient is negative.

USING END BEHAVIOR Describe the degree and leading coefficient of the polynomial function whose graph is shown.



DESCRIBING END BEHAVIOR Describe the end behavior of the graph of the polynomial function by completing these statements: $f(x) \rightarrow ?$ as $x \rightarrow -\infty$ and $f(x) \rightarrow ?$ as $x \rightarrow +\infty$.

28. $f(x) = 10x^4$ 29. $f(x) = -x^6 + 4x^3 - 3x$ 30. $f(x) = -2x^3 + 7x - 4$
31. $f(x) = x^7 + 3x^4 - x^2$ 32. $f(x) = 3x^{10} - 16x$ 33. $f(x) = -6x^5 + 14x^2 + 20$
34. $f(x) = 0.2x^3 - x + 45$ 35. $f(x) = 5x^8 + 8x^7$ 36. $f(x) = -x^{273} + 500x^{271}$

37. ★ **OPEN-ENDED MATH** Write a polynomial function f of degree 5 such that the end behavior of the graph of f is given by $f(x) \rightarrow +\infty$ as $x \rightarrow -\infty$ and $f(x) \rightarrow -\infty$ as $x \rightarrow +\infty$. Then graph the function to verify your answer.

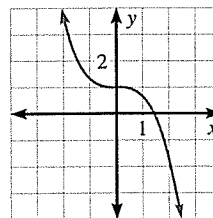
EXAMPLE 5
on p. 340
for Exs. 38–50

GRAPHING POLYNOMIALS Graph the polynomial function.

38. $f(x) = x^3$ 39. $f(x) = -x^4$ 40. $f(x) = x^5 + 3$
41. $f(x) = x^4 - 2$ 42. $f(x) = -x^3 + 5$ 43. $f(x) = x^3 - 5x$
44. $f(x) = -x^4 + 8x$ 45. $f(x) = x^5 + x$ 46. $f(x) = -x^3 + 3x^2 - 2x + 5$
47. $f(x) = x^5 + x^2 - 4$ 48. $f(x) = x^4 - 5x^2 + 6$ 49. $f(x) = -x^4 + 3x^3 - x + 1$

50. ★ **MULTIPLE CHOICE** Which function is represented by the graph shown?

- (A) $f(x) = \frac{1}{3}x^3 + 1$ (B) $f(x) = -\frac{1}{3}x^3 + 1$
(C) $f(x) = \frac{1}{3}x^3 - 1$ (D) $f(x) = -\frac{1}{3}x^3 - 1$



51. **VISUAL THINKING** Suppose $f(x) \rightarrow +\infty$ as $x \rightarrow -\infty$ and $f(x) \rightarrow -\infty$ as $x \rightarrow +\infty$. Describe the end behavior of $g(x) = -f(x)$.

52. ★ **SHORT RESPONSE** A cubic polynomial function f has leading coefficient 2 and constant term -5 . If $f(1) = 0$ and $f(2) = 3$, what is $f(-5)$? Explain how you found your answer.

53. CHALLENGE Let $f(x) = x^3$ and $g(x) = x^3 - 2x^2 + 4x$.

- Copy and complete the table.
- Use the numbers in the table to complete this statement: As $x \rightarrow +\infty$, $\frac{f(x)}{g(x)} \rightarrow \underline{\hspace{1cm}}$.
- Explain how the result from part (b) shows that the functions f and g have the same end behavior as $x \rightarrow +\infty$.

x	$f(x)$	$g(x)$	$\frac{f(x)}{g(x)}$
10	?	?	?
20	?	?	?
50	?	?	?
100	?	?	?
200	?	?	?

PROBLEM SOLVING

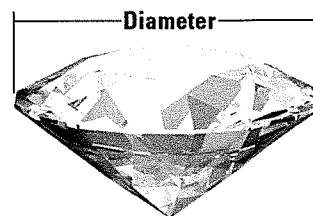
EXAMPLE 6
on p. 340
for Exs. 54–59

- 54. DIAMONDS** The weight of an ideal round-cut diamond can be modeled by

$$w = 0.0071d^3 - 0.090d^2 + 0.48d$$

where w is the diamond's weight (in carats) and d is its diameter (in millimeters). According to the model, what is the weight of a diamond with a diameter of 15 millimeters?

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- 55. SKATEBOARDING** From 1992 to 2003, the number of people in the United States who participated in skateboarding can be modeled by

$$S = -0.0076t^4 + 0.14t^3 - 0.62t^2 + 0.52t + 5.5$$

where S is the number of participants (in millions) and t is the number of years since 1992. Graph the model. Then use the graph to estimate the first year that the number of skateboarding participants was greater than 8 million.

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- 56. MULTIPLE REPRESENTATIONS** From 1987 to 2003, the number of indoor movie screens M in the United States can be modeled by

$$M = -11.0t^3 + 267t^2 - 592t + 21,600$$

where t is the number of years since 1987.

- Classifying a Function** State the degree and type of the function.
- Making a Table** Make a table of values for the function.
- Sketching a Graph** Use your table to graph the function.

- 57. SNOWBOARDING** From 1992 to 2003, the number of people in the United States who participated in snowboarding can be modeled by

$$S = 0.0013t^4 - 0.021t^3 + 0.084t^2 + 0.037t + 1.2$$

where S is the number of participants (in millions) and t is the number of years since 1992. Graph the model. Use the graph to estimate the first year that the number of snowboarding participants was greater than 2 million.



EXAMPLE 5 Use special product patterns

$$\begin{aligned} \text{a. } (3t + 4)(3t - 4) &= (3t)^2 - 4^2 && \text{Sum and difference} \\ &= 9t^2 - 16 \end{aligned}$$

$$\begin{aligned} \text{b. } (8x - 3)^2 &= (8x)^2 - 2(8x)(3) + 3^2 && \text{Square of a binomial} \\ &= 64x^2 - 48x + 9 \end{aligned}$$

$$\begin{aligned} \text{c. } (pq + 5)^3 &= (pq)^3 + 3(pq)^2(5) + 3(pq)(5)^2 + 5^3 && \text{Cube of a binomial} \\ &= p^3q^3 + 15p^2q^2 + 75pq + 125 \end{aligned}$$

**GUIDED PRACTICE** for Examples 3, 4, and 5

Find the product.

3. $(x + 2)(3x^2 - x - 5)$

4. $(a - 5)(a + 2)(a + 6)$

5. $(xy - 4)^3$

EXAMPLE 6 Use polynomial models

PETROLEUM Since 1980, the number W (in thousands) of United States wells producing crude oil and the average daily oil output per well O (in barrels) can be modeled by

$$W = -0.575t^2 + 10.9t + 548 \quad \text{and} \quad O = -0.249t + 15.4$$

where t is the number of years since 1980. Write a model for the average *total* amount T of crude oil produced per day. What was the average total amount of crude oil produced per day in 2000?



Oil refinery in Long Beach, California

DETERMINE SIGNIFICANT DIGITS

When multiplying models, round your result so that its terms have the same number of significant digits as the model with the fewest number of significant digits.

SolutionTo find a model for T , multiply the two given models.

$$\begin{array}{r} -0.575t^2 + 10.9t + 548 \\ \times \quad -0.249t + 15.4 \\ \hline -8.855t^2 + 167.86t + 8439.2 \\ 0.143175t^3 - 2.7141t^2 - 136.452t \\ \hline 0.143175t^3 - 11.5691t^2 + 31.408t + 8439.2 \end{array}$$

- Total daily oil output can be modeled by $T = 0.143t^3 - 11.6t^2 + 31.4t + 8440$ where T is measured in thousands of barrels. By substituting $t = 20$ into the model, you can estimate that the average total amount of crude oil produced per day in 2000 was about 5570 thousand barrels, or 5,570,000 barrels.

**GUIDED PRACTICE** for Example 6

6. **INDUSTRY** The models below give the average depth D (in feet) of new wells drilled and the average cost per foot C (in dollars) of drilling a new well. In both models, t represents the number of years since 1980. Write a model for the average *total* cost T of drilling a new well.

$$D = 109t + 4010 \quad \text{and} \quad C = 0.542t^2 - 7.16t + 79.4$$

5.3 EXERCISES

HOMEWORK KEY

- = WORKED-OUT SOLUTIONS
on p. WS10 for Exs. 11, 21, and 61
- ★ = STANDARDIZED TEST PRACTICE
Exs. 2, 15, 47, 56, and 63

SKILL PRACTICE

- VOCABULARY** When you add or subtract polynomials, you add or subtract the coefficients of ?.
- ★ **WRITING** Explain how a polynomial subtraction problem is equivalent to a polynomial addition problem.

EXAMPLES 1 and 2

on p. 346
for Exs. 3–15

ADDING AND SUBTRACTING POLYNOMIALS Find the sum or difference.

- $(3x^2 - 5) + (7x^2 - 3)$
- $(4y^2 + 9y - 5) - (4y^2 - 5y + 3)$
- $(3s^3 + s) + (4s^3 - 2s^2 + 7s + 10)$
- $(5c^2 + 7c + 1) + (2c^3 - 6c + 8)$
- $(5b - 6b^3 + 2b^4) - (9b^3 + 4b^4 - 7)$
- $(x^4 - x^3 + x^2 - x + 1) + (x + x^4 - 1 - x^2)$
- $(x^2 - 3x + 5) - (-4x^2 + 8x + 9)$
- $(z^2 + 5z - 7) + (5z^2 - 11z - 6)$
- $(2a^2 - 8) - (a^3 + 4a^2 - 12a + 4)$
- $(4t^3 - 11t^2 + 4t) - (-7t^2 - 5t + 8)$
- $(3y^2 - 6y^4 + 5 - 6y) + (5y^4 - 6y^3 + 4y)$
- $(8v^4 - 2v^2 + v - 4) - (3v^3 - 12v^2 + 8v)$
- ★ **MULTIPLE CHOICE** What is the result when $2x^4 - 8x^2 - x + 10$ is subtracted from $8x^4 - 4x^3 - x + 2$?
 - $-6x^4 + 4x^3 - 8x^2 + 8$
 - $6x^4 - 4x^3 + 8x^2 - 8$
 - $10x^4 - 8x^3 - 4x^2 + 12$
 - $6x^4 + 4x^3 - 2x - 8$

EXAMPLE 3

on p. 347
for Exs. 16–25

MULTIPLYING POLYNOMIALS Find the product of the polynomials.

- $x(2x^2 - 5x + 7)$
- $(y - 7)(y + 6)$
- $(w + 4)(w^2 + 6w - 11)$
- $(5c^2 - 4)(2c^2 + c - 3)$
- $(-d^2 + 4d + 3)(3d^2 - 7d + 6)$
- $5x^2(6x + 2)$
- $(3z + 1)(z - 3)$
- $(2a - 3)(a^2 - 10a - 2)$
- $(-x^2 + 4x + 1)(x^2 - 8x + 3)$
- $(3y^2 + 6y - 1)(4y^2 - 11y - 5)$

ERROR ANALYSIS Describe and correct the error in simplifying the expression.

26.

$$\begin{aligned} &(x^2 - 3x + 4) - (x^3 + 7x - 2) \\ &= x^2 - 3x + 4 - x^3 + 7x - 2 \\ &= -x^3 + x^2 + 4x + 2 \end{aligned}$$

27.

$$\begin{aligned} &(2x - 7)^3 = (2x)^3 - 7^3 \\ &= 8x^3 - 343 \end{aligned}$$

EXAMPLE 4

on p. 347
for Exs. 28–37

MULTIPLYING THREE BINOMIALS Find the product of the binomials.

- $(x + 4)(x - 6)(x - 5)$
- $(z - 4)(-z + 2)(z + 8)$
- $(3p + 1)(p + 3)(p + 1)$
- $(2s + 1)(3s - 2)(4s - 3)$
- $(4x - 1)(-2x - 7)(-5x - 4)$
- $(x + 1)(x - 7)(x + 3)$
- $(a - 6)(2a + 5)(a + 1)$
- $(b - 2)(2b - 1)(-b + 1)$
- $(w - 6)(4w - 1)(-3w + 5)$
- $(3q - 8)(-9q + 2)(q - 2)$

64. **CHALLENGE** From 1970 to 2002, the circulation C (in millions) of Sunday newspapers in the United States can be modeled by

$$C = -0.00105t^3 + 0.0281t^2 + 0.465t + 48.8$$

where t is the number of years since 1970. Rewrite C as a function of s , where s is the number of years since 1975.

PA

PENNSYLVANIA MIXED REVIEW

TEST PRACTICE at classzone.com

65. The table shows the total cost y of heating oil. Which equation best represents the total cost of the heating oil as a function of the number of gallons x ?

- (A) $x = 0.67y$ (B) $y = 0.67x$
(C) $x = 1.5y$ (D) $y = 1.5x$

Number of gallons (x)	Total cost (y)
50	\$75
200	\$300
500	\$750

66. A student is making a circle graph of the results of a survey that asked what people's favorite sport is. What central angle should be used for the section representing basketball?

- (A) 35° (B) 105°
(C) 126° (D) 234°

Activity	Number of people
Basketball	350
Soccer	210
Softball or Baseball	200
Other	240

QUIZ for Lessons 5.1–5.3

Evaluate the expression. (p. 330)

1. $3^5 \cdot 3^{-1}$

2. $(2^4)^2$

3. $\left(\frac{2}{3^{-2}}\right)^2$

4. $\left(\frac{3}{5}\right)^{-2}$

Simplify the expression. (p. 330)

5. $(x^4y^{-2})(x^{-3}y^8)$

6. $(a^2b^{-5})^{-3}$

7. $\frac{x^3y^7}{x^{-4}y^0}$

8. $\frac{c^3d^{-2}}{c^5d^{-1}}$

Graph the polynomial function. (p. 337)

9. $g(x) = 2x^3 - 3x + 1$

10. $h(x) = x^4 - 4x + 2$

11. $f(x) = -2x^3 + x^2 - 5$

Perform the indicated operation. (p. 346)

12. $(x^3 + x^2 - 6) - (2x^2 + 4x - 8)$

13. $(-3x^2 + 4x - 10) + (x^2 - 9x + 15)$

14. $(x - 5)(x^2 - 5x + 7)$

15. $(x + 3)(x - 6)(3x - 1)$

16. **NATIONAL DEBT** On July 21, 2004, the national debt of the United States was about \$7,232,000,000,000. The population of the United States at that time was about 294,000,000. Suppose the national debt was divided evenly among everyone in the United States. How much would each person owe? (p. 330)

5.4 Factor and Solve Polynomial Equations

PA M11.D.2.2.2 Factor algebraic expressions, including difference of squares and trinomials. . .

Before

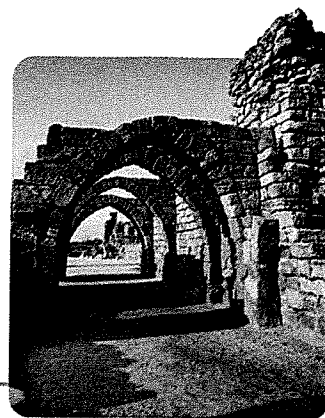
You factored and solved quadratic equations.

Now

You will factor and solve other polynomial equations.

Why?

So you can find dimensions of archaeological ruins, as in Ex. 58.



Key Vocabulary

- factored completely
- factor by grouping
- quadratic form

In Chapter 4, you learned how to factor the following types of quadratic expressions.

Type	Example
General trinomial	$2x^2 - 3x - 20 = (2x + 5)(x - 4)$
Perfect square trinomial	$x^2 + 8x + 16 = (x + 4)^2$
Difference of two squares	$9x^2 - 1 = (3x + 1)(3x - 1)$
Common monomial factor	$8x^2 + 20x = 4x(2x + 5)$

You can also factor polynomials with degree greater than 2. Some of these polynomials can be *factored completely* using techniques learned in Chapter 4.

KEY CONCEPT

For Your Notebook

Factoring Polynomials

Definition

A factorable polynomial with integer coefficients is **factored completely** if it is written as a product of unfactorable polynomials with integer coefficients.

Examples

$2(x + 1)(x - 4)$ and $5x^2(x^2 - 3)$ are factored completely.

$3x(x^2 - 4)$ is *not* factored completely because $x^2 - 4$ can be factored as $(x + 2)(x - 2)$.

EXAMPLE 1

Find a common monomial factor

Factor the polynomial completely.

a. $x^3 + 2x^2 - 15x = x(x^2 + 2x - 15)$ Factor common monomial.
 $= x(x + 5)(x - 3)$ Factor trinomial.

b. $2y^5 - 18y^3 = 2y^3(y^2 - 9)$ Factor common monomial.
 $= 2y^3(y + 3)(y - 3)$ Difference of two squares

c. $4z^4 - 16z^3 + 16z^2 = 4z^2(z^2 - 4z + 4)$ Factor common monomial.
 $= 4z^2(z - 2)^2$ Perfect square trinomial

FACTORIZING PATTERNS In part (b) of Example 1, the special factoring pattern for the difference of two squares is used to factor the expression completely. There are also factoring patterns that you can use to factor the sum or difference of two *cubes*.

KEY CONCEPT

For Your Notebook

Special Factoring Patterns

Sum of Two Cubes

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

Example

$$\begin{aligned} 8x^3 + 27 &= (2x)^3 + 3^3 \\ &= (2x + 3)(4x^2 - 6x + 9) \end{aligned}$$

Difference of Two Cubes

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Example

$$\begin{aligned} 64x^3 - 1 &= (4x)^3 - 1^3 \\ &= (4x - 1)(16x^2 + 4x + 1) \end{aligned}$$

EXAMPLE 2 Factor the sum or difference of two cubes

Factor the polynomial completely.

a. $x^3 + 64 = x^3 + 4^3$

Sum of two cubes

$$= (x + 4)(x^2 - 4x + 16)$$

b. $16z^5 - 250z^2 = 2z^2(8z^3 - 125)$

Factor common monomial.

$$= 2z^2[(2z)^3 - 5^3]$$

Difference of two cubes

$$= 2z^2(2z - 5)(4z^2 + 10z + 25)$$



GUIDED PRACTICE for Examples 1 and 2

Factor the polynomial completely.

1. $x^3 - 7x^2 + 10x$

2. $3y^5 - 75y^3$

3. $16b^5 + 686b^2$

4. $w^3 - 27$

FACTORIZING BY GROUPING For some polynomials, you can **factor by grouping** pairs of terms that have a common monomial factor. The pattern for factoring by grouping is shown below.

$$\begin{aligned} ra + rb + sa + sb &= r(a + b) + s(a + b) \\ &= (r + s)(a + b) \end{aligned}$$

EXAMPLE 3 Factor by grouping

Factor the polynomial $x^3 - 3x^2 - 16x + 48$ completely.

$$x^3 - 3x^2 - 16x + 48 = x^2(x - 3) - 16(x - 3)$$

Factor by grouping.

$$= (x^2 - 16)(x - 3)$$

Distributive property

$$= (x + 4)(x - 4)(x - 3)$$

Difference of two squares

AVOID ERRORS

An expression is not factored completely until *all* factors, such as $x^2 - 16$, cannot be factored further.

QUADRATIC FORM An expression of the form $au^2 + bu + c$, where u is any expression in x , is said to be in **quadratic form**. The factoring techniques you studied in Chapter 4 can sometimes be used to factor such expressions.

EXAMPLE 4 Factor polynomials in quadratic form

IDENTIFY QUADRATIC FORM

The expression $16x^4 - 81$ is in quadratic form because it can be written as $u^2 - 81$ where $u = 4x^2$.

Factor completely: (a) $16x^4 - 81$ and (b) $2p^8 + 10p^5 + 12p^2$.

a. $16x^4 - 81 = (4x^2)^2 - 9^2$

$$= (4x^2 + 9)(4x^2 - 9)$$

$$= (4x^2 + 9)(2x + 3)(2x - 3)$$

Write as difference of two squares.

Difference of two squares

Difference of two squares

b. $2p^8 + 10p^5 + 12p^2 = 2p^2(p^6 + 5p^3 + 6)$

$$= 2p^2(p^3 + 3)(p^3 + 2)$$

Factor common monomial.

Factor trinomial in quadratic form.



GUIDED PRACTICE for Examples 3 and 4

Factor the polynomial completely.

5. $x^3 + 7x^2 - 9x - 63$

6. $16g^4 - 625$

7. $4t^6 - 20t^4 + 24t^2$

SOLVING POLYNOMIAL EQUATIONS In Chapter 4, you learned how to use the zero product property to solve factorable quadratic equations. You can extend this technique to solve some higher-degree polynomial equations.



EXAMPLE 5 Standardized Test Practice

What are the real-number solutions of the equation $3x^5 + 15x = 18x^3$?

(A) 0, 1, 3, 5

(B) -1, 0, 1

(C) 0, 1, $\sqrt{5}$

(D) $-\sqrt{5}$, -1, 0, 1, $\sqrt{5}$

Solution

$$3x^5 + 15x = 18x^3$$

Write original equation.

$$3x^5 - 18x^3 + 15x = 0$$

Write in standard form.

$$3x(x^4 - 6x^2 + 5) = 0$$

Factor common monomial.

$$3x(x^2 - 1)(x^2 - 5) = 0$$

Factor trinomial.

$$3x(x + 1)(x - 1)(x^2 - 5) = 0$$

Difference of two squares

$$x = 0, x = -1, x = 1, x = \sqrt{5}, \text{ or } x = -\sqrt{5}$$

Zero product property

► The correct answer is D. (A) (B) (C) (D)

AVOID ERRORS

Do not divide each side of an equation by a variable or a variable expression, such as $3x$. Doing so will result in the loss of solutions.



GUIDED PRACTICE for Example 5

Find the real-number solutions of the equation.

8. $4x^5 - 40x^3 + 36x = 0$

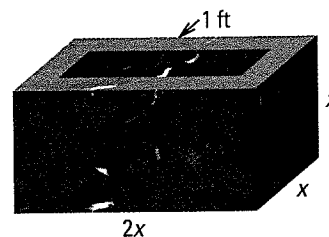
9. $2x^5 + 24x = 14x^3$

10. $-27x^3 + 15x^2 = -6x^4$

EXAMPLE 6 Solve a polynomial equation

CITY PARK You are designing a marble basin that will hold a fountain for a city park. The basin's sides and bottom should be 1 foot thick. Its outer length should be twice its outer width and outer height.

What should the outer dimensions of the basin be if it is to hold 36 cubic feet of water?

**ANOTHER WAY**

For alternative methods to solving the problem in Example 6, turn to page 360 for the **Problem Solving Workshop**.

Solution

Volume (cubic feet)	=	Interior length (feet)	•	Interior width (feet)	•	Interior height (feet)
↓		↓		↓		↓
36	=	$(2x - 2)$	•	$(x - 2)$	•	$(x - 1)$

$36 = (2x - 2)(x - 2)(x - 1)$ **Write equation.**
 $0 = 2x^3 - 8x^2 + 10x - 40$ **Write in standard form.**
 $0 = 2x^2(x - 4) + 10(x - 4)$ **Factor by grouping.**
 $0 = (2x^2 + 10)(x - 4)$ **Distributive property**

► The only real solution is $x = 4$. The basin is 8 ft long, 4 ft wide, and 4 ft high.

**GUIDED PRACTICE** for Example 6

11. **WHAT IF?** In Example 6, what should the basin's dimensions be if it is to hold 40 cubic feet of water and have outer length $6x$, width $3x$, and height x ?

5.4 EXERCISES**HOMEWORK KEY**

- = **WORKED-OUT SOLUTIONS**
on p. WS10 for Exs. 7, 23, and 61
★ = **STANDARDIZED TEST PRACTICE**
Exs. 2, 9, 41, 63, and 64

SKILL PRACTICE

1. **VOCABULARY** The expression $8x^6 + 10x^3 - 3$ is in ? form because it can be written as $2u^2 + 5u - 3$ where $u = 2x^3$.

2. ★ **WRITING** What condition must the factorization of a polynomial satisfy in order for the polynomial to be factored completely?

MONOMIAL FACTORS Factor the polynomial completely.

3. $14x^2 - 21x$

4. $30b^3 - 54b^2$

5. $c^3 + 9c^2 + 18c$

6. $z^3 - 6z^2 - 72z$

7. $3y^5 - 48y^3$

8. $54m^5 + 18m^4 + 9m^3$

9. ★ **MULTIPLE CHOICE** What is the complete factorization of $2x^7 - 32x^3$?

(A) $2x^3(x + 2)(x - 2)(x^2 + 4)$

(B) $2x^3(x^2 + 2)(x^2 - 2)$

(C) $2x^3(x^2 + 4)^2$

(D) $2x^3(x + 2)^2(x - 2)^2$

EXAMPLE 1
on p. 353
for Exs. 3–9

EXAMPLE 2

on p. 354
for Exs. 10–17

SUM OR DIFFERENCE OF CUBES Factor the polynomial completely.

10. $x^3 + 8$ 11. $y^3 - 64$ 12. $27m^3 + 1$ 13. $125n^3 + 216$
14. $27a^3 - 1000$ 15. $8c^3 + 343$ 16. $192w^3 - 3$ 17. $-5z^3 + 320$

EXAMPLE 3

on p. 354
for Exs. 18–23

FACTORING BY GROUPING Factor the polynomial completely.

18. $x^3 + x^2 + x + 1$ 19. $y^3 - 7y^2 + 4y - 28$ 20. $n^3 + 5n^2 - 9n - 45$
21. $3m^3 - m^2 + 9m - 3$ 22. $25s^3 - 100s^2 - s + 4$ 23. $4c^3 + 8c^2 - 9c - 18$

EXAMPLE 4

on p. 355
for Exs. 24–29

QUADRATIC FORM Factor the polynomial completely.

24. $x^4 - 25$ 25. $a^4 + 7a^2 + 6$ 26. $3s^4 - s^2 - 24$
27. $32z^5 - 2z$ 28. $36m^6 + 12m^4 + m^2$ 29. $15x^5 - 72x^3 - 108x$

EXAMPLE 5

on p. 355
for Exs. 30–41

ERROR ANALYSIS Describe and correct the error in finding all real-number solutions.

30.

$$\begin{array}{l} 8x^3 - 27 = 0 \\ (2x + 3)(4x^2 + 6x + 9) = 0 \\ x = -\frac{3}{2} \end{array}$$

31.

$$\begin{array}{l} 3x^3 - 48x = 0 \\ 3x(x^2 - 16) = 0 \\ x^2 - 16 = 0 \\ x = -4 \text{ or } x = 4 \end{array}$$

SOLVING EQUATIONS Find the real-number solutions of the equation.

32. $y^3 - 5y^2 = 0$ 33. $18s^3 = 50s$ 34. $g^3 + 3g^2 - g - 3 = 0$
35. $m^3 + 6m^2 - 4m - 24 = 0$ 36. $4w^4 + 40w^2 - 44 = 0$ 37. $4z^5 = 84z^3$
38. $5b^3 + 15b^2 + 12b = -36$ 39. $x^6 - 4x^4 - 9x^2 + 36 = 0$ 40. $48p^5 = 27p^3$

41. ★ **MULTIPLE CHOICE** What are the real-number solutions of the equation $3x^4 - 27x^2 + 9x = x^3$?

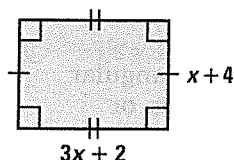
- (A) $-1, 0, 3$ (B) $-3, 0, 3$ (C) $-3, 0, \frac{1}{3}, 3$ (D) $-3, -\frac{1}{3}, 0, 3$

CHOOSING A METHOD Factor the polynomial completely using any method.

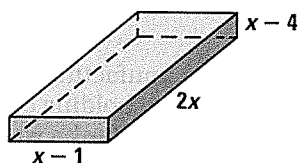
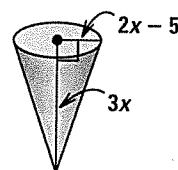
42. $16x^3 - 44x^2 - 42x$ 43. $n^4 - 4n^2 - 60$ 44. $-4b^4 - 500b$
45. $36a^3 - 15a^2 + 84a - 35$ 46. $18c^4 + 57c^3 - 10c^2$ 47. $2d^4 - 13d^2 - 45$
48. $32x^5 - 108x^2$ 49. $8y^6 - 38y^4 - 10y^2$ 50. $z^5 - 3z^4 - 16z + 48$

GEOMETRY Find the possible value(s) of x .

51. Area = 48



52. Volume = 40

53. Volume = 125π **CHOOSING A METHOD** Factor the polynomial completely using any method.

54. $x^3y^6 - 27$ 55. $7ac^2 + bc^2 - 7ad^2 - bd^2$ 56. $x^{2n} - 2x^n + 1$

57. **CHALLENGE** Factor $a^5b^2 - a^2b^4 + 2a^4b - 2ab^3 + a^3 - b^2$ completely.

**PROBLEM SOLVING
WORKSHOP**
LESSON 5.4

Using ALTERNATIVE METHODS

Another Way to Solve Example 6, page 356

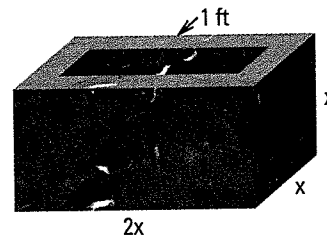


MULTIPLE REPRESENTATIONS In Example 6 on page 356, you solved a polynomial equation by factoring. You can also solve a polynomial equation using a table or a graph.

PROBLEM

CITY PARK You are designing a marble basin that will hold a fountain for a city park. The basin's sides and bottom should be 1 foot thick. Its outer length should be twice its outer width and outer height.

What should the outer dimensions of the basin be if it is to hold 36 cubic feet of water?



METHOD 1

Using a Table One alternative approach is to write a function for the volume of the basin and make a table of values for the function. Using the table, you can find the value of x that makes the volume of the basin 36 cubic feet.

STEP 1 Write the function. From the diagram, you can see that the volume y of water the basin can hold is given by this function:

$$y = (2x - 2)(x - 2)(x - 1)$$

STEP 2 Make a table of values for the function. Use only positive values of x because the basin's dimensions must be positive.

STEP 3 Identify the value of x for which $y = 36$. The table shows that $y = 36$ when $x = 4$.

X	Y1
1	0
2	0
3	8
4	36
5	96
Y1=96	

X	Y1
1	0
2	0
3	8
4	36
5	96
Y1=96	

► The volume of the basin is 36 cubic feet when x is 4 feet. So, the outer dimensions of the basin should be as follows:

$$\text{Length} = 2x = 8 \text{ feet}$$

$$\text{Width} = x = 4 \text{ feet}$$

$$\text{Height} = x = 4 \text{ feet}$$

METHOD 2

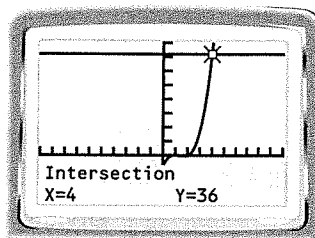
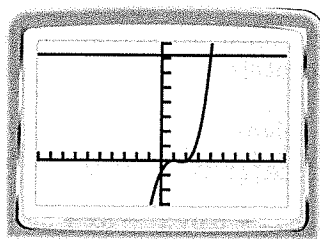
Using a Graph Another approach is to make a graph. You can use the graph to find the value of x that makes the volume of the basin 36 cubic feet.

STEP 1 Write the function. From the diagram, you can see that the volume y of water the basin can hold is given by this function:

$$y = (2x - 2)(x - 2)(x - 1)$$

STEP 2 Graph the equations $y = 36$ and $y = (x - 1)(2x - 2)(x - 2)$. Choose a viewing window that shows the intersection of the graphs.

STEP 3 Identify the coordinates of the intersection point. On a graphing calculator, you can use the *intersect* feature. The intersection point is $(4, 36)$.



► The volume of the basin is 36 cubic feet when x is 4 feet. So, the outer dimensions of the basin should be as follows:

$$\text{Length} = 2x = 8 \text{ feet}$$

$$\text{Width} = x = 4 \text{ feet}$$

$$\text{Height} = x = 4 \text{ feet}$$

PRACTICE

SOLVING EQUATIONS Solve the polynomial equation using a table or using a graph.

1. $x^3 + 4x^2 - 8x = 96$

2. $x^3 - 9x^2 - 14x + 7 = -33$

3. $2x^3 - 11x^2 + 3x + 5 = 59$

4. $x^4 + x^3 - 15x^2 - 8x + 6 = -45$

5. $-x^4 + 2x^3 + 6x^2 + 17x - 4 = 32$

6. $-3x^4 + 4x^3 + 8x^2 + 4x - 11 = 13$

7. $4x^4 - 16x^3 + 29x^2 - 95x = -150$

8. **WHAT IF?** In the problem on page 360, suppose the basin is to hold 200 cubic feet of water. Find the outer dimensions of the basin using a table and using a graph.

9. **PACKAGING** A factory needs a box that has a volume of 1728 cubic inches. The width should be 4 inches less than the height, and the length should be 6 inches greater than the height. Find the dimensions of the box using a table and using a graph.

10. **AGRICULTURE** From 1970 to 2002, the average yearly pineapple consumption P (in pounds) per person in the United States can be modeled by the function

$$P(x) = 0.0000984x^4 - 0.00712x^3 + 0.162x^2 - 1.11x + 12.3$$

where x is the number of years since 1970. In what year was the pineapple consumption about 9.97 pounds per person? Solve the problem using a table and a graph.

FACTOR THEOREM Suppose the remainder is 0 when a polynomial $f(x)$ is divided by $x - k$. Then

$$\frac{f(x)}{x - k} = q(x) + \frac{0}{x - k} = q(x)$$

where $q(x)$ is the quotient polynomial. Therefore, $f(x) = (x - k) \cdot q(x)$, so that $x - k$ is a factor of $f(x)$. This result is summarized by the *factor theorem*.

KEY CONCEPT

For Your Notebook

Factor Theorem

A polynomial $f(x)$ has a factor $x - k$ if and only if $f(k) = 0$.

The factor theorem can be used to solve a variety of problems.

Problem	Example
Given one <i>factor</i> of a polynomial, find the other <i>factors</i> .	See Example 4 below.
Given one <i>zero</i> of a polynomial function, find the other <i>zeros</i> .	See Example 5 on page 365.
Given one <i>solution</i> of a polynomial equation, find the other <i>solutions</i> .	See Example 6 on page 365.

EXAMPLE 4 Factor a polynomial

Factor $f(x) = 3x^3 - 4x^2 - 28x - 16$ completely given that $x + 2$ is a factor.

Solution

Because $x + 2$ is a factor of $f(x)$, you know that $f(-2) = 0$. Use synthetic division to find the other factors.

$$\begin{array}{r|rrrr} -2 & 3 & -4 & -28 & -16 \\ & & -6 & 20 & 16 \\ \hline & 3 & -10 & -8 & 0 \end{array}$$

Use the result to write $f(x)$ as a product of two factors and then factor completely.

$$f(x) = 3x^3 - 4x^2 - 28x - 16 \quad \text{Write original polynomial.}$$

$$= (x + 2)(3x^2 - 10x - 8) \quad \text{Write as a product of two factors.}$$

$$= (x + 2)(3x + 2)(x - 4) \quad \text{Factor trinomial.}$$

AVOID ERRORS

The remainder after using synthetic division should always be zero when you are dividing a polynomial by one of its factors.



GUIDED PRACTICE for Examples 3 and 4

Divide using synthetic division.

3. $(x^3 + 4x^2 - x - 1) \div (x + 3)$

4. $(4x^3 + x^2 - 3x + 7) \div (x - 1)$

Factor the polynomial completely given that $x - 4$ is a factor.

5. $f(x) = x^3 - 6x^2 + 5x + 12$

6. $f(x) = x^3 - x^2 - 22x + 40$

**EXAMPLE 5** Standardized Test Practice

One zero of $f(x) = x^3 - 2x^2 - 23x + 60$ is $x = 3$. What is another zero of f ?

- (A) -5 (B) -4 (C) 2 (D) 5

Solution

Because $f(3) = 0$, $x - 3$ is a factor of $f(x)$. Use synthetic division.

$$\begin{array}{r|rrrr} 3 & 1 & -2 & -23 & 60 \\ & & 3 & 3 & -60 \\ \hline & 1 & 1 & -20 & 0 \end{array}$$

Use the result to write $f(x)$ as a product of two factors. Then factor completely.

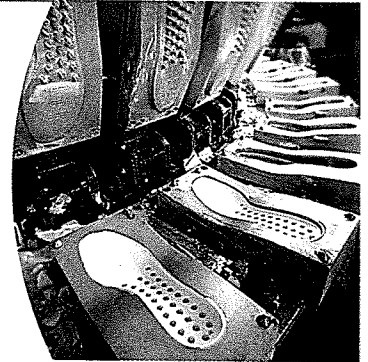
$$f(x) = x^3 - 2x^2 - 23x + 60 = (x - 3)(x^2 + x - 20) = (x - 3)(x + 5)(x - 4)$$

The zeros are 3, -5, and 4.

► The correct answer is A. (A) (B) (C) (D)

EXAMPLE 6 Use a polynomial model

BUSINESS The profit P (in millions of dollars) for a shoe manufacturer can be modeled by $P = -21x^3 + 46x$ where x is the number of shoes produced (in millions). The company now produces 1 million shoes and makes a profit of \$25,000,000, but would like to cut back production. What lesser number of shoes could the company produce and still make the same profit?

**Solution**

$$25 = -21x^3 + 46x \quad \text{Substitute 25 for } P \text{ in } P = -21x^3 + 46x.$$

$$0 = 21x^3 - 46x + 25 \quad \text{Write in standard form.}$$

You know that $x = 1$ is one solution of the equation. This implies that $x - 1$ is a factor of $21x^3 - 46x + 25$. Use synthetic division to find the other factors.

$$\begin{array}{r|rrrr} 1 & 21 & 0 & -46 & 25 \\ & & 21 & 21 & -25 \\ \hline & 21 & 21 & -25 & 0 \end{array}$$

So, $(x - 1)(21x^2 + 21x - 25) = 0$. Use the quadratic formula to find that $x \approx 0.7$ is the other positive solution.

► The company could still make the same profit producing about 700,000 shoes.

**GUIDED PRACTICE** for Examples 5 and 6

Find the other zeros of f given that $f(-2) = 0$.

7. $f(x) = x^3 + 2x^2 - 9x - 18$

8. $f(x) = x^3 + 8x^2 + 5x - 14$

9. **WHAT IF?** In Example 6, how does the answer change if the profit for the shoe manufacturer is modeled by $P = -15x^3 + 40x$?

5.5 EXERCISES

HOMEWORK KEY

- = WORKED-OUT SOLUTIONS
on p. WS10 for Exs. 17, 25, and 43
- ★ = STANDARDIZED TEST PRACTICE
Exs. 2, 35, 39, 44, and 45
- ◆ = MULTIPLE REPRESENTATIONS
Ex. 38

SKILL PRACTICE

EXAMPLES

1 and 2

on pp. 362–363
for Exs. 3–10

EXAMPLE 3

on p. 363
for Exs. 11–20

EXAMPLE 4

on p. 364
for Exs. 21–28

EXAMPLE 5

on p. 365
for Exs. 29–35

1. **VOCABULARY** State the remainder theorem.

2. ★ **WRITING** Synthetic division has been used to divide -3 by $f(x) = x^4 - 5x^2 + 8x - 2$ by $x + 3$. Explain what the colored numbers represent in the division problem.

1	0	-5	8	-2
	-3	9	-12	12
1	-3	4	-4	10

USING LONG DIVISION Divide using polynomial long division.

3. $(x^2 + x - 17) \div (x - 4)$
4. $(3x^2 - 11x - 26) \div (x - 5)$
5. $(x^3 + 3x^2 + 3x + 2) \div (x - 1)$
6. $(8x^2 + 34x - 1) \div (4x - 1)$
7. $(3x^3 + 11x^2 + 4x + 1) \div (x^2 + x)$
8. $(7x^3 + 11x^2 + 7x + 5) \div (x^2 + 1)$
9. $(5x^4 - 2x^3 - 7x^2 - 39) \div (x^2 + 2x - 4)$
10. $(4x^4 + 5x - 4) \div (x^2 - 3x - 2)$

USING SYNTHETIC DIVISION Divide using synthetic division.

11. $(2x^2 - 7x + 10) \div (x - 5)$
12. $(4x^2 - 13x - 5) \div (x - 2)$
13. $(x^2 + 8x + 1) \div (x + 4)$
14. $(x^2 + 9) \div (x - 3)$
15. $(x^3 - 5x^2 - 2) \div (x - 4)$
16. $(x^3 - 4x + 6) \div (x + 3)$
17. $(x^4 - 5x^3 - 8x^2 + 13x - 12) \div (x - 6)$
18. $(x^4 + 4x^3 + 16x - 35) \div (x + 5)$

ERROR ANALYSIS Describe and correct the error in using synthetic division to divide $x^3 - 5x + 3$ by $x - 2$.

19.

2	1	0	-5	3
		2	4	-2
	1	2	-1	1

$\frac{x^3 - 5x + 3}{x - 2} = x^3 + 2x^2 - x + 1$

20.

2	1	-5	3
		2	-6
	1	-3	-3

$\frac{x^3 - 5x + 3}{x - 2} = x^2 - 3x - \frac{3}{x - 2}$

FACTOR Given polynomial $f(x)$ and a factor of $f(x)$, factor $f(x)$ completely.

21. $f(x) = x^3 - 10x^2 + 19x + 30$; $x - 6$
22. $f(x) = x^3 + 6x^2 + 5x - 12$; $x + 4$
23. $f(x) = x^3 - 2x^2 - 40x - 64$; $x - 8$
24. $f(x) = x^3 + 18x^2 + 95x + 150$; $x + 10$
25. $f(x) = x^3 + 2x^2 - 51x + 108$; $x + 9$
26. $f(x) = x^3 - 9x^2 + 8x + 60$; $x + 2$
27. $f(x) = 2x^3 - 15x^2 + 34x - 21$; $x - 1$
28. $f(x) = 3x^3 - 2x^2 - 61x - 20$; $x - 5$

FIND ZEROS Given polynomial function f and a zero of f , find the other zeros.

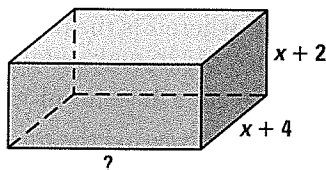
29. $f(x) = x^3 - 2x^2 - 21x - 18$; -3
30. $f(x) = 4x^3 - 25x^2 - 154x + 40$; 10
31. $f(x) = 10x^3 - 81x^2 + 71x + 42$; 7
32. $f(x) = 3x^3 + 34x^2 + 72x - 64$; -4
33. $f(x) = 2x^3 - 10x^2 - 71x - 9$; 9
34. $f(x) = 5x^3 - x^2 - 18x + 8$; -2

35. ★ **MULTIPLE CHOICE** One zero of $f(x) = 4x^3 + 15x^2 - 63x - 54$ is $x = -6$. What is another zero of f ?

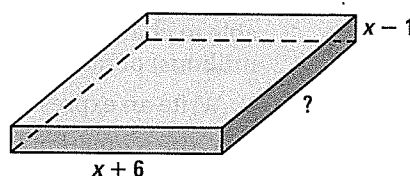
(A) -9 (B) -3 (C) -1 (D) 3

 **GEOMETRY** You are given an expression for the volume of the rectangular prism. Find an expression for the missing dimension.

36. $V = 2x^3 + 17x^2 + 46x + 40$



37. $V = x^3 + 13x^2 + 34x - 48$



38.  **MULTIPLE REPRESENTATIONS** Consider the polynomial function $f(x) = x^3 - 5x^2 - 12x + 36$.

- Zeros of a Function** Given that $f(2) = 0$, find the other zeros of f .
- Factors of an Expression** Based on your results from part (a), what are the factors of the polynomial $x^3 - 5x^2 - 12x + 36$?
- Solutions of an Equation** What are the solutions of the polynomial equation $x^3 - 5x^2 - 12x + 36 = 0$?

39. ★ **MULTIPLE CHOICE** What is the value of k such that $x - 5$ is a factor of $x^3 - x^2 + kx - 30$?

(A) -14 (B) -2 (C) 26 (D) 32

40. **CHALLENGE** It can be shown that $2x - 1$ is a factor of the polynomial function $f(x) = 30x^3 + 7x^2 - 39x + 14$.

- What can you conclude is a zero of f ?
- Use synthetic division to write $f(x)$ in the form $(x - k) \cdot q(x)$.
- Write $f(x)$ as the product of linear factors with integer coefficients.

PROBLEM SOLVING

EXAMPLE 6

on p. 365
for Exs. 41–43

41. **CLOTHING** The profit P (in millions of dollars) for a T-shirt manufacturer can be modeled by $P = -x^3 + 4x^2 + x$ where x is the number of T-shirts produced (in millions). Currently, the company produces 4 million T-shirts and makes a profit of \$4,000,000. What lesser number of T-shirts could the company produce and still make the same profit?

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42. **MP3 PLAYERS** The profit P (in millions of dollars) for a manufacturer of MP3 players can be modeled by $P = -4x^3 + 12x^2 + 16x$ where x is the number of MP3 players produced (in millions). Currently, the company produces 3 million MP3 players and makes a profit of \$48,000,000. What lesser number of MP3 players could the company produce and still make the same profit?

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43. **WOMEN'S BASKETBALL** From 1985 to 2003, the total attendance A (in thousands) at NCAA women's basketball games and the number T of NCAA women's basketball teams can be modeled by

$$A = -1.95x^3 + 70.1x^2 - 188x + 2150 \quad \text{and} \quad T = 14.8x + 725$$

where x is the number of years since 1985. Write a function for the average attendance per team from 1985 to 2003.

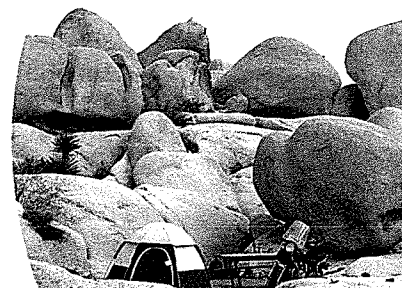
44. **★ EXTENDED RESPONSE** The price p (in dollars) that a radio manufacturer is able to charge for a radio is given by $p = 40 - 4x^2$ where x is the number (in millions) of radios produced. It costs the company \$15 to make a radio.
- Write an expression for the company's total revenue in terms of x .
 - Write a function for the company's profit P by subtracting the total cost to make x radios from the expression in part (a).
 - Currently, the company produces 1.5 million radios and makes a profit of \$24,000,000. Write and solve an equation to find a lesser number of radios that the company could produce and still make the same profit.
 - Do all the solutions in part (c) make sense in this situation? *Explain*.

45. **★ SHORT RESPONSE** Since 1990, overnight stays S and total visits V (both in millions) to national parks can be modeled by

$$S = -0.00722x^4 + 0.176x^3 - 1.40x^2 + 3.39x + 17.6$$

$$V = 3.10x + 256$$

where x is the number of years since 1990. Write a function for the percent of visits to national parks that were overnight stays. *Explain* how you constructed your function.



Joshua Tree National Park, California

46. **CHALLENGE** The profit P (in millions of dollars) for a DVD manufacturer can be modeled by $P = -6x^3 + 72x$ where x is the number of DVDs produced (in millions). Show that 2 million DVDs is the only production level for the company that yields a profit of \$96,000,000.

PA

PENNSYLVANIA MIXED REVIEW

TEST PRACTICE at classzone.com

47. James leaves his home to walk to school. Four minutes later, his friend leaves her home to ride her bike to school. James averages 3 miles per hour and his friend averages 10 miles per hour. James and his friend travel a combined total of 8 miles and arrive at school at the same time. How long did it take James to walk to school?
- (A) 34 min (B) 38 min
(C) 40 min (D) 44 min
48. What are the coordinates of the x -intercept of the graph of $2x + 3y = 15$?
- (A) $(-\frac{15}{2}, 0)$ (B) $(\frac{15}{2}, 0)$
(C) $(0, 5)$ (D) $(13, 0)$

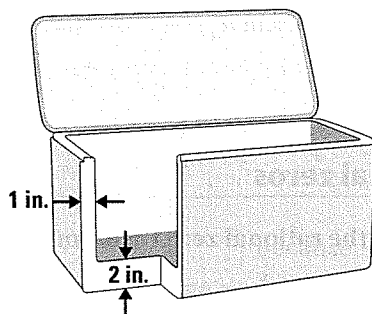


Lessons 5.1–5.5

1. **ASTRONOMY** The average distance between Earth and the sun is 1.64×10^{11} yards. The length of a football field, including the end zones, is 1.20×10^2 yards. About how many football fields stretched end-to-end would it take to reach from Earth to the sun?

A 1.37×10^7
 B 1.37×10^9
 C 1.37×10^{10}
 D 1.37×10^{13}

2. **PRODUCT DESIGN** You are designing a rectangular picnic cooler with length 4 times its width and height 2 times its width. The cooler has insulation that is 1 inch thick on each of the four sides and 2 inches thick on the top and bottom. Let x represent the width of the cooler. What is a polynomial function $C(x)$ in standard form for the volume of the inside of the cooler?



A $C(x) = 8x^3 - 18x^2 + 12x - 2$
 B $C(x) = 8x^3 - 28x^2 + 28x - 8$
 C $C(x) = 8x^3 - 36x^2 + 48x - 16$
 D $C(x) = 8x^3 + 36x^2 + 48x + 16$

3. **END BEHAVIOR** Which polynomial function has degree 4 and end behavior given by $f(x) \rightarrow -\infty$ as $x \rightarrow -\infty$ and $f(x) \rightarrow -\infty$ as $x \rightarrow +\infty$?

A $f(x) = 4x^3 - 4x^2 + x + 5$
 B $f(x) = x^4 - x^3 + 2x^2 - 5x + 2$
 C $f(x) = -x^4 + 5x^2 - x + 20$
 D $f(x) = -4x^6 + x^4 + 4$

4. **PACKAGING DESIGN** A floral shop has a rectangular gift box with a volume of 540 cubic inches. The width of the gift box is 3 inches less than the height, and the length is 15 inches greater than the height. What is the height of the gift box to the nearest tenth of an inch?

A 3.7 inches
 B 6.7 inches
 C 12.0 inches
 D 21.7 inches

5. **MANUFACTURING** The price p (in dollars) that a camera manufacturer is able to charge for a camera is given by $p = 100 - 10x^2$ where x is the number (in millions) of cameras produced. It costs the company \$30 to make a camera. Currently, the company produces 2 million cameras and makes a profit of \$60,000,000. What other number of cameras could the company produce and still make the same profit?

A 1 million
 B 3 million
 C 4 million
 D 5 million

6. **OPEN-ENDED** From 1995 to 2003, the average monthly cell phone bill C (in dollars) for subscribers in the United States can be modeled by the function

$$C = -0.027t^4 + 0.32t^3 - 0.25t^2 - 4.9t + 51$$

where t is the number of years since 1995.

- A. According to this model, what was the average monthly cell phone bill in 2001?
 B. Do you think the model will accurately predict cell phone bills for years beyond 2006? *Explain* your answer.

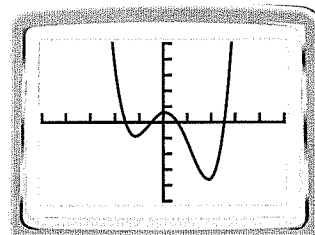
EXAMPLE 3 Find zeros when the leading coefficient is not 1Find all real zeros of $f(x) = 10x^4 - 11x^3 - 42x^2 + 7x + 12$.**Solution**

STEP 1 List the possible rational zeros of f : $\pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{3}{1}, \pm\frac{4}{1}, \pm\frac{6}{1}, \pm\frac{12}{1},$
 $\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{1}{5}, \pm\frac{2}{5}, \pm\frac{3}{5}, \pm\frac{4}{5}, \pm\frac{6}{5}, \pm\frac{12}{5}, \pm\frac{1}{10}, \pm\frac{3}{10}$

STEP 2 Choose reasonable values from the list above to check using the graph of the function. For f , the values

$$x = -\frac{3}{2}, x = -\frac{1}{2}, x = \frac{3}{5}, \text{ and } x = \frac{12}{5}$$

are reasonable based on the graph shown at the right.



STEP 3 Check the values using synthetic division until a zero is found.

$$\begin{array}{r|rrrrrr}
 -\frac{3}{2} & 10 & -11 & -42 & 7 & 12 & \\
 & & -15 & 39 & \frac{9}{2} & -\frac{69}{4} & \\
 \hline
 & 10 & -26 & -3 & \frac{23}{2} & -\frac{21}{4} &
 \end{array}
 \qquad
 \begin{array}{r|rrrrrr}
 -\frac{1}{2} & 10 & -11 & -42 & 7 & 12 & \\
 & & -5 & 8 & 17 & -12 & \\
 \hline
 & 10 & -16 & -34 & 24 & 0 &
 \end{array}$$

$-\frac{1}{2}$ is a zero.

STEP 4 Factor out a binomial using the result of the synthetic division.

$$\begin{aligned}
 f(x) &= \left(x + \frac{1}{2}\right)(10x^3 - 16x^2 - 34x + 24) && \text{Write as a product of factors.} \\
 &= \left(x + \frac{1}{2}\right)(2)(5x^3 - 8x^2 - 17x + 12) && \text{Factor 2 out of the second factor.} \\
 &= (2x + 1)(5x^3 - 8x^2 - 17x + 12) && \text{Multiply the first factor by 2.}
 \end{aligned}$$

STEP 5 Repeat the steps above for $g(x) = 5x^3 - 8x^2 - 17x + 12$. Any zero of g will also be a zero of f . The possible rational zeros of g are:

$$x = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm\frac{1}{5}, \pm\frac{2}{5}, \pm\frac{3}{5}, \pm\frac{4}{5}, \pm\frac{6}{5}, \pm\frac{12}{5}$$

The graph of g shows that $\frac{3}{5}$ may be a zero. Synthetic division shows

$$\text{that } \frac{3}{5} \text{ is a zero and } g(x) = \left(x - \frac{3}{5}\right)(5x^2 - 5x - 20) = (5x - 3)(x^2 - x - 4).$$

It follows that:

$$f(x) = (2x + 1) \cdot g(x) = (2x + 1)(5x - 3)(x^2 - x - 4)$$

STEP 6 Find the remaining zeros of f by solving $x^2 - x - 4 = 0$.

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-4)}}{2(1)} \quad \text{Substitute 1 for } a, -1 \text{ for } b, \text{ and } -4 \text{ for } c \text{ in the quadratic formula.}$$

$$x = \frac{1 \pm \sqrt{17}}{2} \quad \text{Simplify.}$$

► The real zeros of f are $-\frac{1}{2}, \frac{3}{5}, \frac{1 + \sqrt{17}}{2}, \text{ and } \frac{1 - \sqrt{17}}{2}$.

**GUIDED PRACTICE** for Example 3

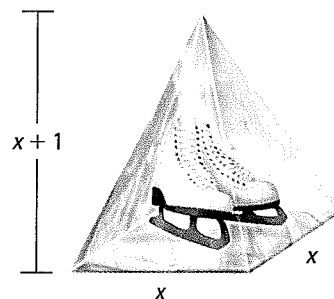
Find all real zeros of the function.

5. $f(x) = 48x^3 + 4x^2 - 20x + 3$

6. $f(x) = 2x^4 + 5x^3 - 18x^2 - 19x + 42$

EXAMPLE 4 Solve a multi-step problem

ICE SCULPTURES Some ice sculptures are made by filling a mold with water and then freezing it. You are making such an ice sculpture for a school dance. It is to be shaped like a pyramid with a height that is 1 foot greater than the length of each side of its square base. The volume of the ice sculpture is 4 cubic feet. What are the dimensions of the mold?

**Solution****STEP 1** Write an equation for the volume of the ice sculpture.

$$\begin{array}{ccccccc} \text{Volume} & = & \frac{1}{3} & \cdot & \text{Area of base} & \cdot & \text{Height} \\ \text{(cubic feet)} & & & & \text{(square feet)} & & \text{(feet)} \\ \downarrow & & & & \downarrow & & \downarrow \\ 4 & = & \frac{1}{3} & \cdot & x^2 & \cdot & (x + 1) \end{array}$$

$$4 = \frac{1}{3}x^2(x + 1) \quad \text{Write equation.}$$

$$12 = x^3 + x^2 \quad \text{Multiply each side by 3 and simplify.}$$

$$0 = x^3 + x^2 - 12 \quad \text{Subtract 12 from each side.}$$

STEP 2 List the possible rational solutions: $\pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{3}{1}, \pm\frac{4}{1}, \pm\frac{6}{1}, \pm\frac{12}{1}$ **STEP 3** Test possible solutions. Only positive x -values make sense.

$$\begin{array}{c|cccc} 1 & 1 & 1 & 0 & -12 \\ & & 1 & 2 & 2 \\ \hline & 1 & 2 & 2 & -10 \end{array} \quad \begin{array}{c|cccc} 2 & 1 & 1 & 0 & -12 \\ & & 2 & 6 & 12 \\ \hline & 1 & 3 & 6 & 0 \end{array}$$

2 is a solution.

STEP 4 Check for other solutions. The other two solutions, which satisfy
$$x^2 + 3x + 6 = 0, \text{ are } x = \frac{-3 \pm i\sqrt{15}}{2} \text{ and can be discarded because they are imaginary numbers.}$$

► The only reasonable solution is $x = 2$. The base of the mold is 2 feet by 2 feet. The height of the mold is $2 + 1 = 3$ feet.

**GUIDED PRACTICE** for Example 4

7. **WHAT IF?** In Example 4, suppose the base of the ice sculpture has sides that are 1 foot longer than the height. The volume of the ice sculpture is 6 cubic feet. What are the dimensions of the mold?

5.6 EXERCISES

HOMEWORK KEY

○ = **WORKED-OUT SOLUTIONS**
on p. WS11 for Exs. 7, 21, and 47

★ = **STANDARDIZED TEST PRACTICE**
Exs. 2, 23, 38, 39, 40, and 50

SKILL PRACTICE

EXAMPLE 1

on p. 370
for Exs. 3–10

1. **VOCABULARY** Copy and complete: If a polynomial function has integer coefficients, then every rational zero of the function has the form $\frac{p}{q}$, where p is a factor of the ? and q is a factor of the ?.

2. ★ **WRITING** Describe a method you can use to shorten the list of possible rational zeros when using the rational zero theorem.

LISTING RATIONAL ZEROS List the possible rational zeros of the function using the rational zero theorem.

3. $f(x) = x^3 - 3x + 28$

4. $g(x) = x^3 - 4x^2 + x - 10$

5. $f(x) = 2x^4 + 6x^3 - 7x + 9$

6. $h(x) = 2x^3 + x^2 - x - 18$

7. $g(x) = 4x^5 + 3x^3 - 2x - 14$

8. $f(x) = 3x^4 + 5x^3 - 3x + 42$

9. $h(x) = 8x^4 + 4x^3 - 10x + 15$

10. $h(x) = 6x^3 - 3x^2 + 12$

EXAMPLE 2

on p. 371
for Exs. 11–18

FINDING REAL ZEROS Find all real zeros of the function.

11. $f(x) = x^3 - 12x^2 + 35x - 24$

12. $f(x) = x^3 - 5x^2 - 22x + 56$

13. $g(x) = x^3 - 31x - 30$

14. $h(x) = x^3 + 8x^2 - 9x - 72$

15. $h(x) = x^4 + 7x^3 + 26x^2 + 44x + 24$

16. $f(x) = x^4 - 2x^3 - 9x^2 + 10x - 24$

17. $f(x) = x^4 + 2x^3 - 9x^2 - 2x + 8$

18. $g(x) = x^4 - 16x^2 - 40x - 25$

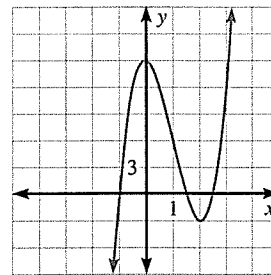
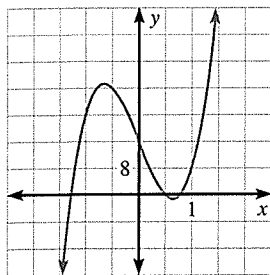
EXAMPLE 3

on p. 372
for Exs. 19–35

ELIMINATING POSSIBLE ZEROS Use the graph to shorten the list of possible rational zeros of the function. Then find all real zeros of the function.

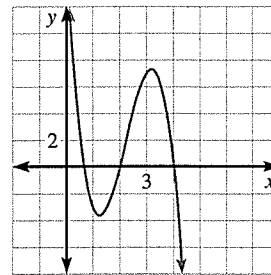
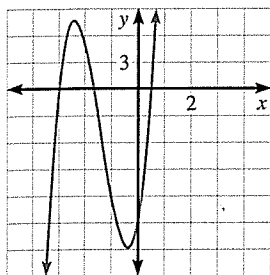
19. $f(x) = 4x^3 - 20x + 16$

20. $f(x) = 4x^3 - 12x^2 - x + 15$



21. $f(x) = 6x^3 + 25x^2 + 16x - 15$

22. $f(x) = -3x^3 + 20x^2 - 36x + 16$



23. ★ **MULTIPLE CHOICE** According to the rational zero theorem, which is *not* a possible zero of the function $f(x) = 2x^4 - 5x^3 + 10x^2 - 9$?

(A) -9 (B) $-\frac{1}{2}$ (C) $\frac{5}{2}$ (D) 3

FINDING REAL ZEROS Find all real zeros of the function.

24. $f(x) = 2x^3 + 2x^2 - 8x - 8$ 25. $g(x) = 2x^3 - 7x^2 + 9$
 26. $h(x) = 2x^3 - 3x^2 - 14x + 15$ 27. $f(x) = 3x^3 + 4x^2 - 35x - 12$
 28. $f(x) = 3x^3 + 19x^2 + 4x - 12$ 29. $g(x) = 2x^3 + 5x^2 - 11x - 14$
 30. $g(x) = 2x^4 + 9x^3 + 5x^2 + 3x - 4$ 31. $h(x) = 2x^4 - x^3 - 7x^2 + 4x - 4$
 32. $h(x) = 3x^4 - 6x^3 - 32x^2 + 35x - 12$ 33. $f(x) = 2x^4 - 9x^3 + 37x - 30$
 34. $f(x) = x^5 - 3x^4 - 5x^3 + 15x^2 + 4x - 12$ 35. $h(x) = 2x^5 + 5x^4 - 3x^3 - 2x^2 - 5x + 3$

ERROR ANALYSIS Describe and correct the error in listing the possible rational zeros of the function.

36. $f(x) = x^3 + 7x^2 + 2x + 14$
 Possible zeros: $1, 2, 7, 14$

37. $f(x) = 6x^3 - 3x^2 + 12x + 5$
 Possible zeros: $\pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{5}, \pm \frac{2}{5}, \pm \frac{3}{5}, \pm \frac{6}{5}$

38. ★ **OPEN-ENDED MATH** Write a polynomial function f that has a leading coefficient of 4 and has 12 possible rational zeros according to the rational zero theorem.

39. ★ **MULTIPLE CHOICE** Which of the following is *not* a zero of the function $f(x) = 40x^5 - 42x^4 - 107x^3 + 107x^2 + 33x - 36$?

(A) $-\frac{3}{2}$ (B) $-\frac{3}{8}$ (C) $\frac{3}{4}$ (D) $\frac{4}{5}$

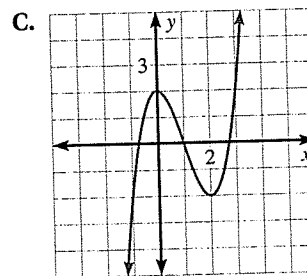
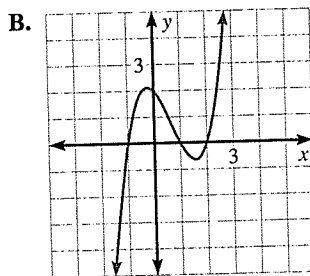
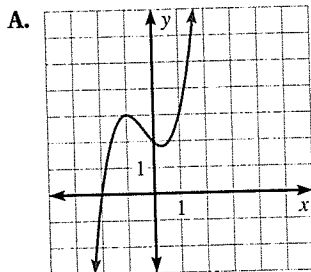
40. ★ **SHORT RESPONSE** Let a_n be the leading coefficient of a polynomial function f and a_0 be the constant term. If a_n has r factors and a_0 has s factors, what is the largest number of possible rational zeros of f that can be generated by the rational zero theorem? *Explain* your reasoning.

MATCHING Find all real zeros of the function. Then match each function with its graph.

41. $f(x) = x^3 - 2x^2 - x + 2$

42. $g(x) = x^3 - 3x^2 + 2$

43. $h(x) = x^3 + x^2 - x + 2$



44. **CHALLENGE** Is it possible for a cubic function to have more than three real zeros? Is it possible for a cubic function to have no real zeros? *Explain*.

EXAMPLE 2 Find the zeros of a polynomial function

Find all zeros of $f(x) = x^5 - 4x^4 + 4x^3 + 10x^2 - 13x - 14$.

Solution

STEP 1 Find the rational zeros of f . Because f is a polynomial function of degree 5, it has 5 zeros. The possible rational zeros are ± 1 , ± 2 , ± 7 , and ± 14 . Using synthetic division, you can determine that -1 is a zero repeated twice and 2 is also a zero.

STEP 2 Write $f(x)$ in factored form. Dividing $f(x)$ by its known factors $x + 1$, $x + 1$, and $x - 2$ gives a quotient of $x^2 - 4x + 7$. Therefore:

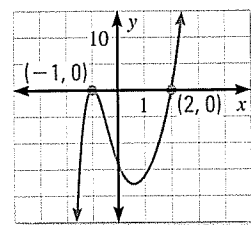
$$f(x) = (x + 1)^2(x - 2)(x^2 - 4x + 7)$$

STEP 3 Find the complex zeros of f . Use the quadratic formula to factor the trinomial into linear factors.

$$f(x) = (x + 1)^2(x - 2)[x - (2 + i\sqrt{3})][x - (2 - i\sqrt{3})]$$

► The zeros of f are -1 , -1 , 2 , $2 + i\sqrt{3}$, and $2 - i\sqrt{3}$.

BEHAVIOR NEAR ZEROS The graph of f in Example 2 is shown at the right. Note that only the *real* zeros appear as x -intercepts. Also note that the graph is tangent to the x -axis at the repeated zero $x = -1$, but crosses the x -axis at the zero $x = 2$. This concept can be generalized as follows:



- When a factor $x - k$ of a function f is raised to an odd power, the graph of f crosses the x -axis at $x = k$.
- When a factor $x - k$ of a function f is raised to an even power, the graph of f is tangent to the x -axis at $x = k$.



GUIDED PRACTICE for Example 2

Find all zeros of the polynomial function.

3. $f(x) = x^3 + 7x^2 + 15x + 9$

4. $f(x) = x^5 - 2x^4 + 8x^2 - 13x + 6$

REVIEW COMPLEX NUMBERS

For help with complex conjugates, see p. 278.

COMPLEX CONJUGATES Also in Example 2, notice that the zeros $2 + i\sqrt{3}$ and $2 - i\sqrt{3}$ are complex conjugates. This illustrates the first theorem given below. A similar result applies to irrational zeros of polynomial functions, as shown in the second theorem below.

KEY CONCEPT

For Your Notebook

Complex Conjugates Theorem

If f is a polynomial function with real coefficients, and $a + bi$ is an imaginary zero of f , then $a - bi$ is also a zero of f .

Irrational Conjugates Theorem

Suppose f is a polynomial function with rational coefficients, and a and b are rational numbers such that $\sqrt{b}$ is irrational. If $a + \sqrt{b}$ is a zero of f , then $a - \sqrt{b}$ is also a zero of f .

EXAMPLE 3 Use zeros to write a polynomial function

Write a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and 3 and $2 + \sqrt{5}$ as zeros.

Solution

Because the coefficients are rational and $2 + \sqrt{5}$ is a zero, $2 - \sqrt{5}$ must also be a zero by the irrational conjugates theorem. Use the three zeros and the factor theorem to write $f(x)$ as a product of three factors.

$$f(x) = (x - 3)[x - (2 + \sqrt{5})][x - (2 - \sqrt{5})] \quad \text{Write } f(x) \text{ in factored form.}$$

$$= (x - 3)[(x - 2) - \sqrt{5}][(x - 2) + \sqrt{5}] \quad \text{Regroup terms.}$$

$$= (x - 3)(x - 2)^2 - 5] \quad \text{Multiply.}$$

$$= (x - 3)(x^2 - 4x + 4) - 5] \quad \text{Expand binomial.}$$

$$= (x - 3)(x^2 - 4x - 1) \quad \text{Simplify.}$$

$$= x^3 - 4x^2 - x - 3x^2 + 12x + 3 \quad \text{Multiply.}$$

$$= x^3 - 7x^2 + 11x + 3 \quad \text{Combine like terms.}$$

CHECK You can check this result by evaluating $f(x)$ at each of its three zeros.

$$f(3) = 3^3 - 7(3)^2 + 11(3) + 3 = 27 - 63 + 33 + 3 = 0 \quad \checkmark$$

$$\begin{aligned} f(2 + \sqrt{5}) &= (2 + \sqrt{5})^3 - 7(2 + \sqrt{5})^2 + 11(2 + \sqrt{5}) + 3 \\ &= 38 + 17\sqrt{5} - 63 - 28\sqrt{5} + 22 + 11\sqrt{5} + 3 \\ &= 0 \quad \checkmark \end{aligned}$$

Since $f(2 + \sqrt{5}) = 0$, by the irrational conjugates theorem $f(2 - \sqrt{5}) = 0$. $\checkmark$

**GUIDED PRACTICE** for Example 3

Write a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and the given zeros.

5. $-1, 2, 4$

6. $4, 1 + \sqrt{5}$

7. $2, 2i, 4 - \sqrt{6}$

8. $3, 3 - i$

DESCARTES' RULE OF SIGNS French mathematician René Descartes (1596–1650) found the following relationship between the coefficients of a polynomial function and the number of positive and negative zeros of the function.

KEY CONCEPT*For Your Notebook***Descartes' Rule of Signs**

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ be a polynomial function with real coefficients.

- The number of *positive real zeros* of f is equal to the number of changes in sign of the coefficients of $f(x)$ or is less than this by an even number.
- The number of *negative real zeros* of f is equal to the number of changes in sign of the coefficients of $f(-x)$ or is less than this by an even number.

EXAMPLE 2

on p. 380
for Exs. 10–19

FINDING ZEROS Find all zeros of the polynomial function.

10. $f(x) = x^4 - 6x^3 + 7x^2 + 6x - 8$

11. $f(x) = x^4 + 5x^3 - 7x^2 - 29x + 30$

12. $g(x) = x^4 - 9x^2 - 4x + 12$

13. $h(x) = x^3 + 5x^2 - 4x - 20$

14. $f(x) = x^4 + 15x^2 - 16$

15. $f(x) = x^4 + x^3 + 2x^2 + 4x - 8$

16. $h(x) = x^4 + 4x^3 + 7x^2 + 16x + 12$

17. $g(x) = x^4 - 2x^3 - 3x^2 + 2x + 2$

18. $g(x) = 4x^4 + 4x^3 - 11x^2 - 12x - 3$

19. $h(x) = 2x^4 + 13x^3 + 19x^2 - 10x - 24$

EXAMPLE 3

on p. 381
for Exs. 20–32

WRITING POLYNOMIAL FUNCTIONS Write a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and the given zeros.

20. 1, 2, 3

21. -2, 1, 3

22. -5, -1, 2

23. -3, 1, 6

24. 2, $-i$, i

25. $3i$, $2 - i$

26. -1, 2, $-3i$

27. 5, 5, $4 + i$

28. 4, $-\sqrt{5}$, $\sqrt{5}$

29. -4, 1, $2 - \sqrt{6}$

30. -2, -1, 2, 3, $\sqrt{11}$

31. 3, $4 + 2i$, $1 + \sqrt{7}$

32. **ERROR ANALYSIS** Describe and correct the error in writing a polynomial function with rational coefficients and zeros 2 and $1 + i$.

$$\begin{aligned} f(x) &= (x - 2)[x - (1 + i)] \\ &= x(x - 1 - i) - 2(x - 1 - i) \end{aligned}$$

33. **★ OPEN-ENDED MATH** Write a polynomial function of degree 5 with zeros 1, 2, and $-i$.

$$\begin{aligned} &= x^2 - x - ix - 2x + 2 + 2i \\ &= x^2 - (3 + i)x + (2 + 2i) \end{aligned}$$

**EXAMPLE 4**

on p. 382
for Exs. 34–41

CLASSIFYING ZEROS Determine the possible numbers of positive real zeros, negative real zeros, and imaginary zeros for the function.

34. $f(x) = x^4 - x^2 - 6$

35. $g(x) = -x^3 + 5x^2 + 12$

36. $g(x) = x^3 - 4x^2 + 8x + 7$

37. $h(x) = x^5 - 2x^3 - x^2 + 6x + 5$

38. $h(x) = x^5 - 3x^3 + 8x - 10$

39. $f(x) = x^5 + 7x^4 - 4x^3 - 3x^2 + 9x - 15$

40. $g(x) = x^6 + x^5 - 3x^4 + x^3 + 5x^2 + 9x - 18$

41. $f(x) = x^7 + 4x^4 - 10x + 25$

EXAMPLE 5

on p. 382
for Exs. 42–49

APPROXIMATING ZEROS Use a graphing calculator to graph the function. Then use the *zero* (or *root*) feature to approximate the real zeros of the function.

42. $f(x) = x^3 - x^2 - 8x + 5$

43. $f(x) = -x^4 - 4x^2 + x + 8$

44. $g(x) = x^3 - 3x^2 + x + 6$

45. $h(x) = x^4 - 5x - 3$

46. $h(x) = 3x^3 - x^2 - 5x + 3$

47. $g(x) = x^4 - x^3 + 2x^2 - 6x - 3$

48. $f(x) = 2x^6 + x^4 + 31x^2 - 35$

49. $g(x) = x^5 - 16x^3 - 3x^2 + 42x + 30$

50. **REASONING** Two zeros of $f(x) = x^3 - 6x^2 - 16x + 96$ are 4 and -4. Explain why the third zero must also be a real number.

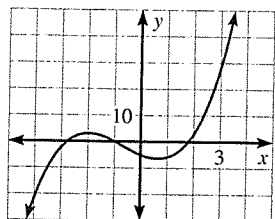
51. **★ SHORT RESPONSE** Describe the possible numbers of positive real, negative real, and imaginary zeros for a cubic function with rational coefficients.

52. **★ MULTIPLE CHOICE** Which is *not* a possible classification of the zeros of $f(x) = x^5 - 4x^3 + 6x^2 + 12x - 6$ according to Descartes' rule of signs?

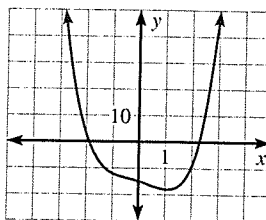
- (A) 3 positive real zeros, 2 negative real zeros, and 0 imaginary zeros
- (B) 3 positive real zeros, 0 negative real zeros, and 2 imaginary zeros
- (C) 1 positive real zero, 4 negative real zeros, and 0 imaginary zeros
- (D) 1 positive real zero, 2 negative real zeros, and 2 imaginary zeros

CLASSIFYING ZEROS Determine the numbers of positive real zeros, negative real zeros, and imaginary zeros for the function with the given degree and graph. Explain your reasoning.

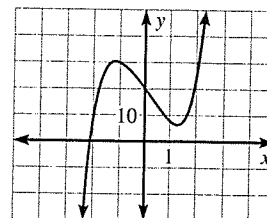
53. Degree: 3



54. Degree: 4



55. Degree: 5



CHALLENGE Show that the given number is a zero of the given function but that the conjugate of the number is *not* a zero.

56. $f(x) = x^3 - 2x^2 + 2x + 5i; 2 - i$

57. $g(x) = x^3 + 2x^2 + 2i - 2; -1 + i$

58. Explain why the results of Exercises 56 and 57 do not contradict the complex conjugate theorem on page 380.

PROBLEM SOLVING

EXAMPLE 6
on p. 383
for Exs. 59–62

59. **BUSINESS** For the 12 years that a grocery store has been open, its annual revenue R (in millions of dollars) can be modeled by the function

$$R = 0.0001(-t^4 + 12t^3 - 77t^2 + 600t + 13,650)$$

where t is the number of years since the store opened. In which year(s) was the revenue \$1.5 million?

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60. **ENVIRONMENT** From 1990 to 2003, the number N of inland lakes in Michigan infested with zebra mussels can be modeled by the function

$$N = -0.028t^4 + 0.59t^3 - 2.5t^2 + 8.3t - 2.5$$

where t is the number of years since 1990. In which year did the number of infested inland lakes first reach 120?

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Pipe clogged with zebra mussels

61. **PHYSIOLOGY** A study group found that a person's score S on a step-climbing exercise test was related to his or her amount of hemoglobin x (in grams per 100 milliliters of blood) by this function:

$$S = -0.015x^3 + 0.6x^2 - 2.4x + 19$$

Given that the normal range of hemoglobin is 12–18 grams per 100 milliliters of blood, what is the most likely amount of hemoglobin for a person who scores 75?

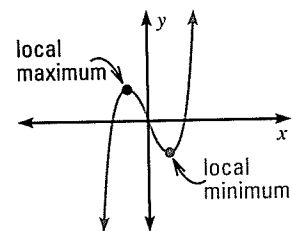
62. **POPULATION** From 1890 to 2000, the American Indian, Eskimo, and Aleut population P (in thousands) can be modeled by the function

$$P = 0.0035t^3 - 0.235t^2 + 4.87t + 243$$

where t is the number of years since 1890. In which year did the population first reach 722,000?

TURNING POINTS Another important characteristic of graphs of polynomial functions is that they have *turning points* corresponding to local maximum and minimum values.

- The y -coordinate of a turning point is a **local maximum** of the function if the point is higher than all nearby points.
- The y -coordinate of a turning point is a **local minimum** of the function if the point is lower than all nearby points.



KEY CONCEPT

For Your Notebook

Turning Points of Polynomial Functions

1. The graph of every polynomial function of degree n has *at most* $n - 1$ turning points.
2. If a polynomial function has n distinct real zeros, then its graph has *exactly* $n - 1$ turning points.

EXAMPLE 2 Find turning points

Graph the function. Identify the x -intercepts and the points where the local maximums and local minimums occur.

a. $f(x) = x^3 - 3x^2 + 6$

b. $g(x) = x^4 - 6x^3 + 3x^2 + 10x - 3$

Solution

- a. Use a graphing calculator to graph the function.

Notice that the graph of f has one x -intercept and two turning points.

You can use the graphing calculator's *zero*, *maximum*, and *minimum* features to approximate the coordinates of the points.

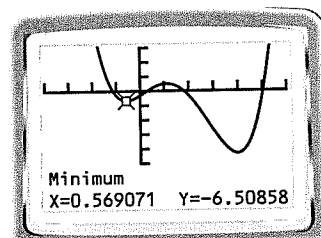
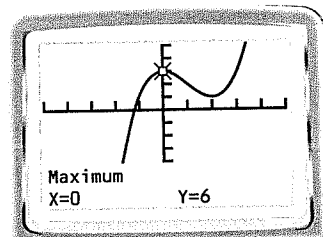
- The x -intercept of the graph is $x \approx -1.20$. The function has a local maximum at $(0, 6)$ and a local minimum at $(2, 2)$.

- b. Use a graphing calculator to graph the function.

Notice that the graph of g has four x -intercepts and three turning points.

You can use the graphing calculator's *zero*, *maximum*, and *minimum* features to approximate the coordinates of the points.

- The x -intercepts of the graph are $x \approx -1.14$, $x \approx 0.29$, $x \approx 1.82$, and $x \approx 5.03$. The function has a local maximum at $(1.11, 5.11)$ and local minimums at $(-0.57, -6.51)$ and $(3.96, -43.04)$.



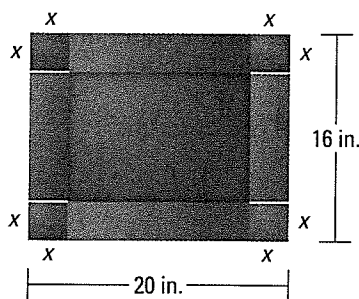
FIND MAXIMUMS AND MINIMUMS

For help with using the *maximum* and *minimum* features of a graphing calculator, see p. 244.

**EXAMPLE 3** Maximize a polynomial model

ARTS AND CRAFTS You are making a rectangular box out of a 16-inch-by-20-inch piece of cardboard. The box will be formed by making the cuts shown in the diagram and folding up the sides. You want the box to have the greatest volume possible.

- How long should you make the cuts?
- What is the maximum volume?
- What will the dimensions of the finished box be?

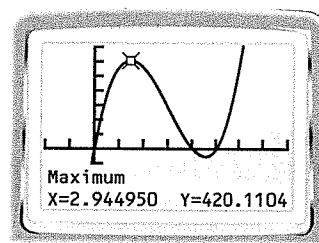
**Solution**

Write a verbal model for the volume. Then write a function.

Volume (cubic inches)	=	Length (inches)	•	Width (inches)	•	Height (inches)
↓		↓		↓		↓
V	=	$(20 - 2x)$	•	$(16 - 2x)$	•	x
	=	$(320 - 72x + 4x^2)x$				
		Multiply binomials.				
	=	$4x^3 - 72x^2 + 320x$				
		Write in standard form.				

To find the maximum volume, graph the volume function on a graphing calculator, as shown at the right. Consider only the interval $0 < x < 8$ because this describes the physical restrictions on the size of the flaps.

From the graph, you can see that the maximum volume is about 420 and occurs when $x \approx 2.94$.



► You should make the cuts about 3 inches long.

The maximum volume is about 420 cubic inches. The dimensions of the box with this volume will be about $x = 3$ inches by $x = 10$ inches by $x = 14$ inches.

**GUIDED PRACTICE** for Examples 1, 2, and 3

Graph the function. Identify the x -intercepts and the points where the local maximums and local minimums occur.

- $f(x) = 0.25(x + 2)(x - 1)(x - 3)$
- $g(x) = 2(x - 1)^2(x - 4)$
- $h(x) = 0.5x^3 + x^2 - x + 2$
- $f(x) = x^4 + 3x^3 - x^2 - 4x - 5$
- WHAT IF?** In Example 3, how do the answers change if the piece of cardboard is 10 inches by 15 inches?

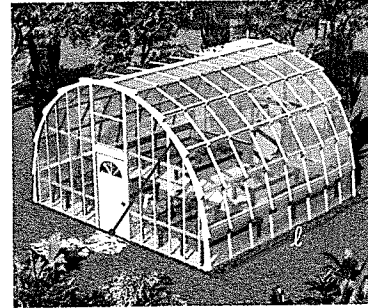
41. **SWIMMING** For a swimmer doing the breaststroke, the function

$$S = -241t^7 + 1060t^6 - 1870t^5 + 1650t^4 - 737t^3 + 144t^2 - 2.43t$$

models the swimmer's speed S (in meters per second) during one complete stroke, where t is the number of seconds since the start of the stroke. Graph the function. According to the model, at what time during the stroke is the swimmer going the fastest?

42. **MULTIPLE REPRESENTATIONS** You have 600 square feet of material for building a greenhouse that is shaped like half a cylinder.

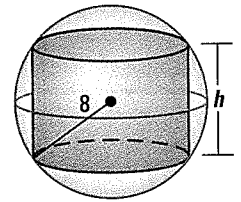
- Writing an Expression** The surface area S of the greenhouse is given by $S = \pi r^2 + \pi r\ell$. Substitute 600 for S and then write an expression for ℓ in terms of r .
- Writing a Function** The volume V of the greenhouse is given by $V = \frac{1}{2}\pi r^2\ell$. Write an equation that gives V as a polynomial function of r alone.
- Graphing a Function** Graph the volume function from part (b). What are the dimensions r and ℓ that maximize the volume of the greenhouse? What is the maximum volume?



43. **★ EXTENDED RESPONSE** From 1960 to 2001, the number of students S (in thousands) enrolled in public schools in the United States can be modeled by $S = 1.64x^3 - 102x^2 + 1710x + 36,300$ where x is the number of years since 1960.

- Graph the function.
- Identify any turning points on the domain $0 \leq x \leq 41$. What real-life meaning do these points have?
- What is the range of the function?

44. **CHALLENGE** A cylinder is inscribed in a sphere of radius 8. Write an equation for the volume of the cylinder as a function of h . Find the value of h that maximizes the volume of the inscribed cylinder. What is the maximum volume of the cylinder?



PA

PENNSYLVANIA MIXED REVIEW

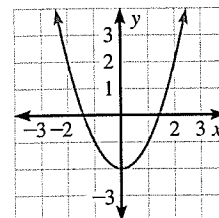
TEST PRACTICE at classzone.com

45. A painter is repainting a spherical section of a sculpture. Which measure would be most useful in determining the amount of paint the painter needs to buy?

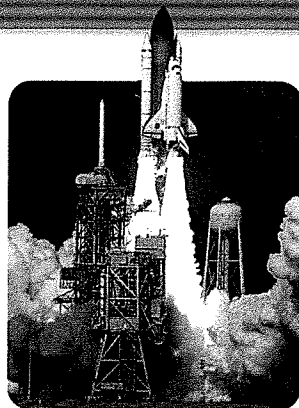
- | | |
|------------|-------------------|
| (A) Radius | (B) Circumference |
| (C) Volume | (D) Surface area |

46. Which equation is the parent function of the graph represented?

- | | |
|---------------|---------------|
| (A) $y = x$ | (B) $y = x $ |
| (C) $y = x^2$ | (D) $y = x^3$ |



5.9 Write Polynomial Functions and Models



Before

You wrote linear and quadratic functions.

Now

You will write higher-degree polynomial functions.

Why?

So you can model launch speed, as in Example 4.

Key Vocabulary
• finite differences

You know that two points determine a line and that three points determine a parabola. In Example 1, you will see that four points determine the graph of a cubic function.

EXAMPLE 1 Write a cubic function

Write the cubic function whose graph is shown.

Solution

STEP 1 Use the three given x -intercepts to write the function in factored form.

$$f(x) = a(x + 4)(x - 1)(x - 3)$$

STEP 2 Find the value of a by substituting the coordinates of the fourth point.

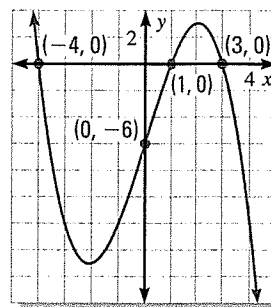
$$-6 = a(0 + 4)(0 - 1)(0 - 3)$$

$$-6 = 12a$$

$$-\frac{1}{2} = a$$

► The function is $f(x) = -\frac{1}{2}(x + 4)(x - 1)(x - 3)$.

CHECK Check the end behavior of f . The degree of f is odd and $a < 0$. So $f(x) \rightarrow +\infty$ as $x \rightarrow -\infty$ and $f(x) \rightarrow -\infty$ as $x \rightarrow +\infty$, which matches the graph. ✓



FINITE DIFFERENCES In Example 1, you found a function given its graph. Functions can also be written from a set of data using *finite differences*.

When the x -values in a data set are equally spaced, the differences of consecutive y -values are called **finite differences**. For example, some finite differences for the function $f(x) = x^2$ are shown below.

$f(1)$	$f(2)$	$f(3)$	$f(4)$	
1	4	9	16	← Values of $f(x)$ for equally-spaced x -values
	$4 - 1 = 3$	$9 - 4 = 5$	$16 - 9 = 7$	← Finite differences

The finite differences above are called *first-order differences*. You can also calculate higher-order differences, as shown in the next example.

CUBIC REGRESSION In Examples 1 and 3, you found a cubic model that *exactly* fits a set of data points. In many real-life situations, you cannot find a simple model to fit data points exactly. Instead, you can use the *regression* feature of a graphing calculator to find an *n*th-degree polynomial model that best fits the data.



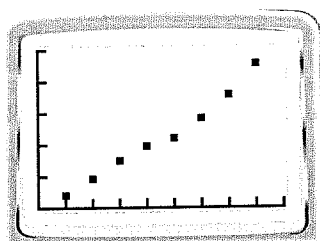
EXAMPLE 4 Solve a multi-step problem

SPACE EXPLORATION The table shows the typical speed y (in feet per second) of a space shuttle x seconds after launch. Find a polynomial model for the data. Use the model to predict the time when the shuttle's speed reaches 4400 feet per second, at which point its booster rockets detach.

x	10	20	30	40	50	60	70	80
y	202.4	463.3	748.2	979.3	1186.3	1421.3	1795.4	2283.5

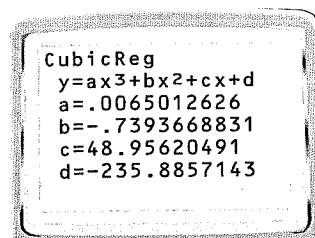
Solution

STEP 1 Enter the data into a graphing calculator and make a scatter plot. The points suggest a cubic model.

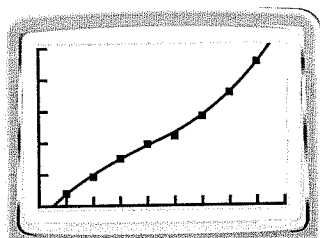


STEP 2 Use cubic regression to obtain this polynomial model:

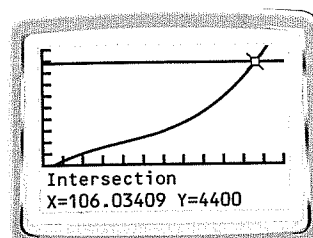
$$y = 0.00650x^3 - 0.739x^2 + 49.0x - 236$$



STEP 3 Check the model by graphing it and the data in the same viewing window.



STEP 4 Graph the model and $y = 4400$ in the same viewing window. Use the *intersect* feature.



ANOTHER WAY

You can also find the value of x for which $y = 4400$ by subtracting 4400 from the right side of the cubic model, graphing the resulting function, and using the *zero* feature to find the graph's x -intercept.

► The booster rockets detach about 106 seconds after launch.

Animated Algebra at classzone.com



GUIDED PRACTICE for Example 4

Use a graphing calculator to find a polynomial function that fits the data.

5.

x	1	2	3	4	5	6
$f(x)$	5	13	17	11	11	56

6.

x	0	2	4	6	8	10
$f(x)$	8	0	15	69	98	87

5.9 EXERCISES

HOMEWORK KEY

○ = **WORKED-OUT SOLUTIONS**
on p. WS11 for Exs. 9, 15, and 27
★ = **STANDARDIZED TEST PRACTICE**
Exs. 2, 10, 22, 23, and 28

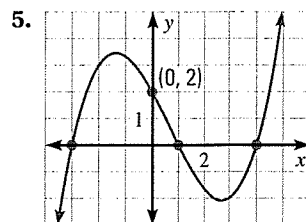
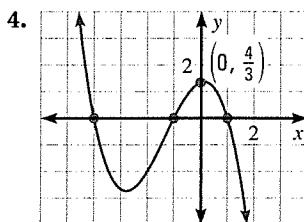
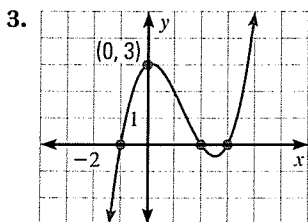
SKILL PRACTICE

- VOCABULARY** Copy and complete: When the x -values in a data set are equally spaced, the differences of consecutive y -values are called ? .
- ★ **WRITING** Describe first-order differences and second-order differences.

EXAMPLE 1

on p. 393
for Exs. 3–11

WRITING CUBIC FUNCTIONS Write the cubic function whose graph is shown.



CUBIC MODELS Write a cubic function whose graph passes through the points.

- $(-3, 0), (-1, 10), (0, 0), (4, 0)$
- $(-2, 0), (-1, 0), (0, -8), (2, 0)$
- $(-3, 0), (1, 0), (3, 2), (4, 0)$
9. $(-5, 0), (0, 0), (1, -12), (6, 0)$

- ★ **MULTIPLE CHOICE** Which cubic function's graph passes through the points $(-3, 0), (-1, 0), (3, 0)$, and $(0, 3)$?

- Ⓐ $f(x) = (x - 3)(x + 3)(x - 1)$ Ⓑ $f(x) = -\frac{1}{3}(x - 3)(x + 3)(x + 1)$
Ⓒ $f(x) = -2(x - 3)(x + 3)(x - 1)$ Ⓓ $f(x) = (x - 3)(x + 3)(x + 1)$

- ERROR ANALYSIS** A student tried to write a cubic function whose graph has x -intercepts $-1, 2$, and 5 , and passes through $(1, 3)$. Describe and correct the error in the student's calculation of the leading coefficient a .

$$\begin{aligned} 1 &= a(3 + 1)(3 - 2)(3 - 5) \\ 1 &= -8a \\ -\frac{1}{8} &= a \end{aligned}$$



EXAMPLE 2

on p. 394
for Exs. 12–17

FINDING FINITE DIFFERENCES Show that the n th-order differences for the given function of degree n are nonzero and constant.

- $f(x) = 5x^3 - 10$
- $f(x) = -2x^2 + 5x$
- $f(x) = x^4 - 3x^2 + 2$
15. $f(x) = 4x^2 - 9x + 2$
- $f(x) = x^3 - 4x^2 - x + 1$
- $f(x) = 2x^5 - 3x^2 + x$

EXAMPLE 3

on p. 395
for Exs. 18–21

FINDING A MODEL Use finite differences and a system of equations to find a polynomial function that fits the data in the table.

18.

x	1	2	3	4	5	6
$f(x)$	0	-3	-8	-15	-24	-35

19.

x	1	2	3	4	5	6
$f(x)$	11	14	9	-4	-25	-54

20.

x	1	2	3	4	5	6
$f(x)$	-12	-14	-10	6	40	98

21.

x	1	2	3	4	5	6
$f(x)$	5	14	27	41	53	60

5

CHAPTER REVIEW

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- Multi-Language Glossary
- Vocabulary practice

REVIEW KEY VOCABULARY

- scientific notation, p. 331
- polynomial, p. 337
- polynomial function, p. 337
- leading coefficient, p. 337
- degree, p. 337
- constant term, p. 337
- standard form of a polynomial function, p. 337
- synthetic substitution, p. 338
- end behavior, p. 339
- factored completely, p. 353
- factor by grouping, p. 354
- quadratic form, p. 355
- polynomial long division, p. 362
- synthetic division, p. 363
- repeated solution, p. 379
- local maximum, p. 388
- local minimum, p. 388
- finite differences, p. 393

VOCABULARY EXERCISES

1. Copy and complete: At each of its turning points, the graph of a polynomial function has a(n) ? or a(n) ?.
2. **WRITING** Explain how you can tell whether a solution of a polynomial equation is a repeated solution when the equation is written in factored form.
3. **WRITING** Explain how you can tell whether a number is expressed in scientific notation.
4. Let f be a fourth-degree polynomial function with four distinct real zeros. How many turning points does the graph of f have?

REVIEW EXAMPLES AND EXERCISES

Use the review examples and exercises below to check your understanding of the concepts you have learned in each lesson of Chapter 5.

5.1

Use Properties of Exponents

pp. 330–335

EXAMPLE

Simplify the expression.

$$\begin{aligned}(x^2y^3)^3x^4 &= (x^2)^3(y^3)^3x^4 && \text{Power of a product property} \\ &= x^6y^9x^4 && \text{Power of a power property} \\ &= x^{6+4}y^9 && \text{Product of powers property} \\ &= x^{10}y^9 && \text{Simplify exponent.}\end{aligned}$$

EXERCISES

Evaluate or simplify the expression. Tell which properties of exponents you used.

5. $2^2 \cdot 2^5$
6. $(3^2)^{-3}(3^3)$
7. $(x^{-2}y^5)^2$
8. $(3x^4y^{-2})^{-3}$
9. $\left(\frac{3}{4}\right)^{-2}$
10. $\frac{8 \times 10^7}{2 \times 10^3}$
11. $\left(\frac{x^2}{y^{-2}}\right)^{-4}$
12. $\frac{2x^{-6}y^5}{16x^3y^{-2}}$

EXAMPLES
1, 2, 3, and 4
on pp. 330–332
for Exs. 5–12

5.2 Evaluate and Graph Polynomial Functions

pp. 337–344

EXAMPLE

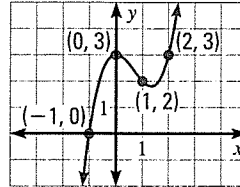
Graph the polynomial function $f(x) = x^3 - 2x^2 + 3$.

Make a table of values.

x	-2	-1	0	1	2	3
$f(x)$	-13	0	3	2	3	12

Plot the points, connect the points with a smooth curve, and check the end behavior.

The degree is odd and the leading coefficient is positive, so $f(x) \rightarrow -\infty$ as $x \rightarrow -\infty$ and $f(x) \rightarrow +\infty$ as $x \rightarrow +\infty$.



EXERCISES

Graph the polynomial function.

13. $f(x) = -x^4$

14. $f(x) = x^3 - 4$

15. $f(x) = x^3 + 2x + 3$

16. **FISH CONSUMPTION** From 1990 to 2002, the amount of fish F (in millions of pounds) caught for human consumption in the United States can be modeled by

$$F = -0.907t^4 + 28.0t^3 - 258t^2 + 902t + 12,700$$

where t is the number of years since 1990. Graph the function. Use the graph to estimate the year when the amount of fish caught first was greater than 14.5 billion pounds.

EXAMPLES

5 and 6

on p. 340

for Exs. 13–16

5.3 Add, Subtract, and Multiply Polynomials

pp. 346–352

EXAMPLE

Perform the indicated operation.

$$\begin{aligned} \text{a. } (3x^3 - 6x^2 - 7x + 5) + (x^3 + 8x + 3) &= 3x^3 + x^3 - 6x^2 - 7x + 8x + 5 + 3 \\ &= 4x^3 - 6x^2 + x + 8 \end{aligned}$$

$$\begin{aligned} \text{b. } (x - 4)(2x^2 - 7x + 5) &= (x - 4)2x^2 - (x - 4)7x + (x - 4)5 \\ &= 2x^3 - 8x^2 - 7x^2 + 28x + 5x - 20 \\ &= 2x^3 - 15x^2 + 33x - 20 \end{aligned}$$

EXERCISES

Perform the indicated operation.

17. $(5x^3 - x + 3) + (x^3 - 9x^2 + 4x)$

18. $(x^3 + 4x^2 - 5x) - (4x^3 + x^2 - 7)$

19. $(x - 6)(5x^2 + x - 8)$

20. $(x - 4)(x + 7)(5x - 1)$

EXAMPLES

1, 2, 4, and 5

on pp. 346–348

for Exs. 17–20

5

CHAPTER REVIEW

5.4 Factor and Solve Polynomial Equations

pp. 353–359

EXAMPLE

Factor the polynomial completely.

- | | |
|---|-------------------------------------|
| a. $x^3 + 125 = x^3 + 5^3 = (x + 5)(x^2 - 5x + 25)$ | Sum of two cubes |
| b. $x^3 + 5x^2 - 9x - 45 = x^2(x + 5) - 9(x + 5)$ | Factor by grouping. |
| $= (x^2 - 9)(x + 5)$ | Distributive property |
| $= (x + 3)(x - 3)(x + 5)$ | Difference of two squares |
| c. $3x^6 + 12x^4 - 96x^2 = 3x^2(x^4 + 4x^2 - 32)$ | Factor common monomial. |
| $= 3x^2(x^2 - 4)(x^2 + 8)$ | Factor trinomial in quadratic form. |
| $= 3x^2(x + 2)(x - 2)(x^2 + 8)$ | Difference of two squares |

EXERCISES

Factor the polynomial completely.

21. $64x^3 - 8$ 22. $2x^5 - 12x^3 + 10x$ 23. $2x^3 - 7x^2 - 8x + 28$
24. **SCULPTURE** You have 240 cubic inches of clay with which to make a sculpture shaped as a rectangular prism. You want the width to be 4 inches less than the length and the height to be 2 inches more than 3 times the length. What should the dimensions of the sculpture be?

EXAMPLES
2, 3, 4, and 6
on pp. 354–356
for Exs. 21–24

5.5 Apply the Remainder and Factor Theorems

pp. 362–368

EXAMPLE

Divide $f(x) = 4x^4 + 29x^3 + 4x^2 - 14x + 37$ by $x + 7$.

Rewrite the divisor in the form $x - k$. Because $x + 7 = x - (-7)$, $k = -7$.

$$\begin{array}{r|rrrrr}
 -7 & 4 & 29 & 4 & -14 & 37 \\
 & & -28 & -7 & 21 & -49 \\
 \hline
 & 4 & 1 & -3 & 7 & -12
 \end{array}$$

$$\text{So, } \frac{4x^4 + 29x^3 + 4x^2 - 14x + 37}{x + 7} = 4x^3 + x^2 - 3x + 7 - \frac{12}{x + 7}.$$

EXERCISES

Divide.

- | | |
|---|--|
| 25. $(x^3 - 3x^2 - x - 10) \div (x^2 + 3x - 1)$ | 26. $(4x^4 - 17x^2 + 9x - 18) \div (2x^2 - 2)$ |
| 27. $(2x^3 - 11x^2 + 13x - 44) \div (x - 5)$ | 28. $(5x^4 + 2x^2 - 15x + 10) \div (x + 2)$ |

Given polynomial $f(x)$ and a factor of $f(x)$, factor $f(x)$ completely.

- | | |
|---|---|
| 29. $f(x) = x^3 - 5x^2 - 2x + 24$; $x + 2$ | 30. $f(x) = x^3 - 11x^2 + 14x + 80$; $x - 8$ |
| 31. $f(x) = 9x^3 - 9x^2 - 4x + 4$; $x - 1$ | 32. $f(x) = 2x^3 + 7x^2 - 33x - 18$; $x + 6$ |

EXAMPLES
1, 3, and 4
on pp. 362–364
for Exs. 25–32

5.6 Find Rational Zeros

pp. 370–377

EXAMPLEFind all real zeros of $f(x) = x^3 + 6x^2 + 5x - 12$.The leading coefficient is 1 and the constant term is -12 .Possible rational zeros: $x = \pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{3}{1}, \pm\frac{4}{1}, \pm\frac{6}{1}, \pm\frac{12}{1}$ Test these zeros using synthetic division. Test $x = 1$:

$$\begin{array}{r|rrrr}
 1 & 1 & 6 & 5 & -12 \\
 & & 1 & 7 & 12 \\
 \hline
 & 1 & 7 & 12 & 0
 \end{array}
 \quad \leftarrow 1 \text{ is a zero.}$$

You can write $f(x) = (x - 1)(x^2 + 7x + 12)$. Factor the trinomial.

$$f(x) = (x - 1)(x^2 + 7x + 12) = (x - 1)(x + 3)(x + 4)$$

The zeros of f are 1, -3 , and -4 .

EXAMPLES
2 and 3
 on pp. 371–372
 for Exs. 33–34

EXERCISES

Find all real zeros of the function.

33. $f(x) = x^3 - 4x^2 - 11x + 30$

34. $f(x) = 2x^4 - x^3 - 42x^2 + 16x + 160$

5.7 Apply the Fundamental Theorem of Algebra

pp. 379–386

EXAMPLEWrite a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and -4 and $5 + \sqrt{2}$ as zeros.Because $5 + \sqrt{2}$ is a zero, $5 - \sqrt{2}$ must also be a zero.

$$f(x) = (x + 4)[x - (5 + \sqrt{2})][x - (5 - \sqrt{2})] \quad \text{Write } f(x) \text{ in factored form.}$$

$$= (x + 4)[(x - 5) - \sqrt{2}][(x - 5) + \sqrt{2}] \quad \text{Regroup terms.}$$

$$= (x + 4)[(x - 5)^2 - 2] \quad \text{Multiply.}$$

$$= x^3 - 6x^2 - 17x + 92 \quad \text{Multiply.}$$

EXERCISESWrite a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and the given zeros.

35. $-4, 1, 5$

36. $-1, -1, 6, 3i$

37. $2, 7, 3 - \sqrt{5}$

38. **ECONOMICS** For the 15 years that a computer store has been open, its annual revenue R (in millions of dollars) can be modeled by

$$R = -0.0040t^4 + 0.088t^3 - 0.36t^2 - 0.55t + 5.8$$

where t is the number of years since the store opened. In what year was the revenue first greater than \$7 million?

EXAMPLES
3 and 6
 on pp. 381–383
 for Exs. 35–38

OPEN-ENDED

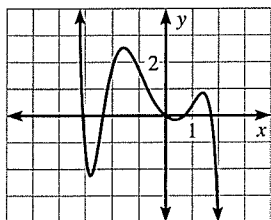
1. Since 1960, the number of voters y (in millions) in United States federal elections can be modeled by the function

$$y = -0.0006x^3 + 0.0383x^2 + 0.383x + 68.6$$

where x is the number of years since 1960.

- According to the model, how many more people voted in 1980 than in 1960?
- Did more people vote in 1980 or in 2000?

2. Consider the graph below.



- What does the graph of the polynomial function tell you about the sign of the leading coefficient?
 - What does the graph tell you about the degree of function and the number of real zeroes?
3. Since 1970, the average fuel efficiency E (in miles per gallon) for all vehicles in the United States can be modeled by the function

$$E = -0.0007t^3 + 0.0278t^2 - 0.0843t + 12.0$$

where t is the number of years since 1970.

- Use a graphing calculator to graph the function, and identify any turning points on the interval $0 \leq t \leq 30$.
- What real-life meaning does a turning point have in this situation?

4. Consider the function below.

$$f(x) = x^4 - 4x^2 - 5$$

- Can you use the rational zero theorem to find the zeros of the function?
 - Find all roots of the function.
5. From 1990 to 2003, the number of CD singles (in millions) sold in the United States can be modeled by the polynomial function
- $$y = 0.014x^5 - 0.40x^4 + 3.8x^3 - 13x^2 + 15x + 1.2$$
- where x is the number of years since 1990.
- Use a graphing calculator to graph the function on the domain $0 \leq x \leq 13$. According to the model, in which year were the most CD singles sold?
 - Do you think that sales will continue to follow the model indefinitely? *Explain* your reasoning.
6. Consider the following tables.

x	1	2	3	4	5	6
$f(x)$	4	9	26	57	104	169

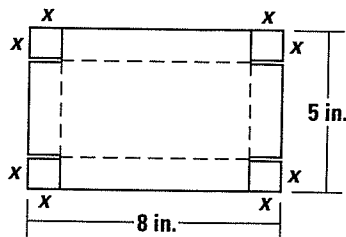
x	1	2	3	4	5	6
$g(x)$	-2	-2	12	52	130	258

- Use finite differences and a system of equations to find a polynomial function that fits the data in each table.
- Find $f(x) + g(x)$. *Explain* all of your steps.



OPEN-ENDED

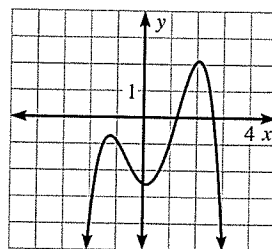
7. You are making an open box to hold paper clips out of a piece of cardboard that is 5 inches by 8 inches. The box will be formed by making the cuts shown in the diagram and folding up the sides. You want the box to have the greatest volume possible.



- Use a graphing calculator to find how long you should make the cuts. *Explain your reasoning.*
 - What is the maximum volume of the box?
 - What will the dimensions of the finished box be?
8. From 1980 to 2002, the number of hospitals H in the United States and the average number of hospital beds B in each hospital can be modeled by
- $$H = -58.7t + 7070 \quad \text{and} \quad B = 0.0066t^3 - 0.192t^2 - 0.174t + 196$$
- where t is the number of years since 1980.
- Write a model for the total number of hospital beds in U.S. hospitals.
 - According to the model, how many beds were in U.S. hospitals in 1995?
 - How does the model change if you want to find the number of hospital beds in thousands? *Explain your reasoning.*

MULTIPLE CHOICE

9. Which expression is equivalent to $\frac{x^2y}{z^4}$?
- $\frac{z^{-4}y^0}{x^{-2}}$
 - $xyz \cdot \frac{x}{z^{-3}}$
 - $(x^{-1}y^2z^2)^2(x^{-1}y^1z^2)^{-4}$
 - $\frac{(x^2yz)^3}{x^4y^2z^7}$
10. What are all the real solutions of the equation $x^4 = 125x$?
- 0
 - 0, 5, -5
 - 0, 5
 - 0, 5i, -5i
11. Which polynomial function has -1, 3, and -4i as zeros?
- $f(x) = x^4 - 2x^3 + 13x^2 - 32x - 48$
 - $f(x) = x^4 + 2x^3 + 13x^2 + 32x - 48$
 - $f(x) = x^4 - 2x^3 - 19x^2 + 32x + 48$
 - $f(x) = x^4 + 2x^3 + 19x^2 - 32x + 48$
12. How many *real* zeros does the function $f(x) = 2x^4 + 3x^2 - 1$ have?
- 0 real zeros
 - 1 real zero
 - 2 real zeros
 - 4 real zeros
13. Evaluate the expression $\left(\frac{3}{2}\right)^{-2}$.
- $\frac{4}{9}$
 - $\frac{9}{4}$
 - $\sqrt{\frac{2}{3}}$
 - $\sqrt{\frac{3}{2}}$
14. The graph of a quartic function is shown. How many imaginary zeros does the function have?



- 0 imaginary zeros
- 1 imaginary zero
- 2 imaginary zeros
- 4 imaginary zeros

6

Rational Exponents and Radical Functions

PA

M11.A.2.2.1

M11.A.2.2.1

M11.D.1.1.3

M11.D.1.1.3

- 6.1 Evaluate n th Roots and Use Rational Exponents
- 6.2 Apply Properties of Rational Exponents
- 6.3 Perform Function Operations and Composition
- 6.4 Use Inverse Functions
- 6.5 Graph Square Root and Cube Root Functions
- 6.6 Solve Radical Equations

Before

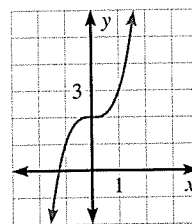
In previous chapters, you learned the following skills, which you'll use in Chapter 6: simplifying expressions involving exponents, rewriting equations, and graphing polynomial functions.

Prerequisite Skills

VOCABULARY CHECK

Copy and complete the statement.

- The **square roots** of 81 are ? and ?.
- In the expression 2^5 , the **exponent** is ?.
- For the polynomial function whose graph is shown, the sign of the **leading coefficient** is ?.



SKILLS CHECK

Simplify the expression. (Review p. 330 for 6.2.)

4. $\frac{5x^2y}{15x^3y^{-1}}$

5. $\frac{32x^{-3}y^4}{24x^{-3}y^{-2}} \cdot \frac{3x}{9y}$

6. $(2x^5y^{-3})^{-3}$

Solve the equation for y . (Review p. 26 for 6.4.)

7. $-2x - 5y = 10$

8. $x - \frac{1}{3}y = -1$

9. $8x - 4xy = 3$

Graph the polynomial function. (Review p. 337 for 6.5.)

10. $f(x) = x^3 - 4x + 6$

11. $f(x) = -x^5 + 7x^2 + 2$

12. $f(x) = x^4 - 4x^2 + x$